Contents

MARKING SCHEME

LEAVING CERTIFICATE EXAMINATION 2006

MATHEMATICS – HIGHER LEVEL – PAPER 1

GENERAL GUIDELINES FOR EXAMINERS – PAPER 1

1. Penalties of three types are applied to candidates' work as follows:
   - Blunders - mathematical errors/omissions (-3)
   - Slips - numerical errors (-1)
   - Misreadings (provided task is not oversimplified) (-1).

   Frequently occurring errors to which these penalties must be applied are listed in the scheme. They are labelled: B1, B2, B3,…, S1, S2,…, M1, M2,…etc. These lists are not exhaustive.

2. When awarding attempt marks, e.g. Att(3), note that
   - any *correct, relevant* step in a part of a question merits at least the attempt mark for that part
   - if deductions result in a mark which is lower than the attempt mark, then the attempt mark must be awarded
   - a mark between zero and the attempt mark is never awarded.

3. Worthless work is awarded zero marks. Some examples of such work are listed in the scheme and they are labelled as W1, W2,…etc.

4. The phrase "hit or miss" means that partial marks are not awarded – the candidate receives all of the relevant marks or none.

5. The phrase "and stops" means that no more work is shown by the candidate.

6. Special notes relating to the marking of a particular part of a question are indicated by an asterisk. These notes immediately follow the box containing the relevant solution.

7. The sample solutions for each question are not intended to be exhaustive lists – there may be other correct solutions. Any examiner unsure of the validity of the approach adopted by a particular candidate to a particular question should contact his/her advising examiner.

8. Unless otherwise indicated in the scheme, accept the best of two or more attempts – even when attempts have been cancelled.

9. The *same* error in the *same* section of a question is penalised *once* only.

10. Particular cases, verifications and answers derived from diagrams (unless requested) qualify for attempt marks at most.

11. A serious blunder, omission or misreading results in the attempt mark at most.

12. Do not penalise the use of a comma for a decimal point, e.g. €5.50 may be written as €5,50.

QUESTION 1

Part (a)	**10 marks**	**Att 3**
Part (b)	**20 (5, 5, 5, 5) marks**	**Att (2, 2, 2, 2)**
Part (c)	**20 (5, 10, 5) marks**	**Att (2, 3, 2)**

Part (a) **10 marks** **Att 3**

1. **(a)** Find the real number a such that for all $x \neq 9$,

$$\frac{x-9}{\sqrt{x}-3} = \sqrt{x} + a.$$

Part 1(a) **10 marks** **Att 3**

1 (a)

$$\frac{x-9}{\sqrt{x}-3} \cdot \frac{\sqrt{x}+3}{\sqrt{x}+3} = \frac{(x-9).(\sqrt{x}+3)}{(x-9)} = \sqrt{x}+3 \qquad \Rightarrow a = 3$$

or

1(a)

$$\frac{x-9}{\sqrt{x}-3} = \frac{(\sqrt{x})^2-(3)^2}{(\sqrt{x}-3)} = \frac{(\sqrt{x}-3)(\sqrt{x}+3)}{(\sqrt{x}-3)}$$

$$= \sqrt{x}+3$$

$$\Rightarrow a = 3$$

or

1(a)

$$\frac{x-9}{\sqrt{x}-3} = \sqrt{x} + a$$

True for all $x \neq 9$, so let $x = 0$ (or any other chosen value):

$$\therefore \quad \frac{0-9}{0-3} = 0 + a$$

$$\Rightarrow \quad a = 3$$

or

1(a)

$$\frac{x-9}{\sqrt{x}-3} = \sqrt{x} + a.$$

$$(x-9) = (\sqrt{x}-3).(\sqrt{x}+a)$$

$$(x-9) = x - 3\sqrt{x} + a\sqrt{x} - 3a$$

$$x + (0)\sqrt{x} + (-9) = x + (a-3)\sqrt{x} + (-3a)$$

Equating Coefficients:

(i) $0 = a - 3$, $a = 3$ **(ii)** $-9 = -3a$, $a = 3$

Blunders (-3)

B1 Indices

B2 $(\sqrt{x}-3)(\sqrt{x}+3) \neq x-9$

B3 In equating coefficients (not like to like)

B4 Squares both sides initially

Attempts

A1 No conjugate

A2 If oversimplified (no surd in answer) or x in answer

Part 1(b)	**20 (5, 5, 5, 5) marks**	**Att (2, 2, 2, 2)**

1 (b) $f(x)=3x^3+mx^2-17x+n$, where m and n are constants.
Given that $x-3$ and $x+2$ are factors of $f(x)$, find the value of m and the value of n.

f(3)	**5 marks**	**Att 2**
f(-2)	**5 marks**	**Att 2**
Equations	**5 marks**	**Att 2**
Solving	**5 marks**	**Att 2**

1 (b)

$$f(x)=3x^3+mx^2-17x+n$$

$(x-3)$ a factor $\Rightarrow f(3)=0$

$$f(3)=3(3)^3+m(3)^2-17(3)+n=0$$

$$9m+n=-30 \ldots\ldots\ldots\ldots\ldots(i)$$

$(x+2)$ factor $\Rightarrow f(-2)=0$

$$f(-2)=3(-2)^3+m(-2)^2-17(-2)+n=0$$

$$4m+n=-10 \ldots\ldots\ldots\ldots\ldots(ii)$$

(i) : $9m+n=-30$
(ii) : $4m+n=-10$
$5m=-20$
$m=-4$

(ii) : $4m+n=-10$
$-16+n=-10$
$n=6$

or

1 (b)

$$\begin{array}{r}3x+(m+3)\\ x^2-x-6\overline{)3x^3+mx^2-17x+n}\\ \underline{3x^3-3x^2-18x}\\ (m+3)x^2+x+n\\ \underline{(m+3)x^2-(m+3)x-6(m+3)}\\ (m+4)x+(6m+n+18)=(0)x+(0)\end{array}$$

(i) : $m+4=0$
$m=-4$

(ii) $6m+n+18=0$
$-24+n+18=0$
$n=6$

or

1 (b) $(x+2)(x-3)=x^2-x-6$

Other factor must be linear $=(ax+b)$

$$(x^2-x-6).(ax+b)=3x^3+mx^2-17x+n$$

$$ax^3-ax^2-6ax+bx^2-bx-6b=3x^3+mx^2-17x+n$$

$$ax^3+(-a+b)x^2+(-6a-b)x+(-6b)=3x^3+mx^2+(-17)x+n$$

Equating Coefficients

(i) $a=3$ (ii) $-a+b=m$ (iii) $-6a-b=-17$ (iv) $-6b=n$

(ctd...)

(iii) : $-6a - b = -17$ $-18 - b = -17$ $-1 = b$ (ii) $m = -a + b$ $= -3 - 1$ $= -4$ (iv) $n = -6b = -6(-1) = 6$ $m = -4$ $n = 6$

Blunders (-3)
B1 Deduction root from factor
B2 Indices
B3 2nd value not found (having found 1st)
B4 In equating coefficients (not like to like)

Slips (-1)
S1 Not changing sign when subtracting in division

Part 1(c)	**20 (5, 10, 5) marks**	**Att (2, 3, 2)**

1 (c) $x^2 - t$ is a factor of $x^3 - px^2 - qx + r$. **(i)** Show that $pq = r$. **(ii)** Express the roots of $x^3 - px^2 - qx + r = 0$ in terms of p and q.

Division	**5 marks**	**Att 2**
(i) Show	**10 marks**	**Att 3**
(ii) Express	**5 marks**	**Att 2**

1 (c) (i)

$$\begin{array}{r} x - p \\ x^2 - t \,\overline{)\, x^3 - px^2 - qx + r} \\ x^3 \qquad -tx \\ \hline -px^2 + (t-q)x + r \\ -px^2 \qquad + pt \\ \hline (t-q)x + (r - pt) = (0)x + (0) \end{array}$$

Equating Coefficients:

(i) : $t - q = 0$
$t = q$

(ii) $r - pt = 0$
$r = pt$
$r = pq$

1 (c) (ii)

$f(x) = (x^2 - t)(x - p) = 0$

$\Rightarrow x^2 - t = 0$ or $x - p = 0$

$x^2 = t$ $x = p$

$x = \pm\sqrt{t}$

$x = \pm\sqrt{q}$

Roots: $\{\pm\sqrt{q}, p\}$

or

1 (c) (i) $f(x) = (x^2 - t)\left(x - \frac{r}{t}\right)$

$$= x^3 - \frac{r}{t}x^2 - tx + r$$

$$= x^3 - px^2 - qx + r$$

Equating coefficients:

(i) $\frac{r}{t} = p$

$r = pt$

$r = pq$

(ii) $t = q$

1 (c) (ii) *as above*

*Remainder $\neq 0$ in division

Blunders (-3)

B1 Indices
B2 In equating coefficients (not like to like)
B3 Root from factor
B4 Root omitted
B5 Roots not in p and q
B6 Show not in required form

Slips (-1)

S1 Not changing sign when subtracting in division.

Attempts

A1 Any attempt at division
A2 Other factor not linear

QUESTION 2

Part (a)	**15 (5, 5, 5)marks**	**Att (2, 2, 2)**
Part (b)	**20 (5, 5, 10) marks**	**Att (2, 2, 3)**
Part (c)	**15 (5, 5, 5) marks**	**Att (2, 2, 2)**

Part (a) **15 (5, 5, 5)marks** **Att (2, 2, 2)**

2. **(a)** Solve the simultaneous equations

$$y = 2x - 5$$

$$x^2 + xy = 2.$$

Part (a) **(i) Substitution** **5 marks** **Att 2**

(ii) 1st Variable **5 marks** **Att 2**

(iii) 2nd Variable **5 marks** **Att 2**

2 (a)

$$(i): y = 2x - 5$$

$$(ii): x^2 + xy = 2$$

$$(ii): x^2 + xy - 2 = 0$$

$$x^2 + x(2x-5) - 2 = 0$$

$$3x^2 - 5x - 2 = 0$$

$$(3x+1).(x-2) = 0$$

$$3x + 1 = 0 \quad \text{or} \quad x - 2 = 0$$

$$x = -\frac{1}{3} \qquad x = 2$$

$$(i): y = 2x - 5$$

$$x = -\frac{1}{3} \;:\; y = 2\left(-\frac{1}{3}\right) - 5 = -\frac{17}{3} \Rightarrow \left(-\frac{1}{3}, -\frac{17}{3}\right)$$

$$x = 2 \;:\; y = 2x - 5 = 2(2) - 5 = -1 \Rightarrow (2, -1)$$

Blunders (-3)

B1 Indices

B2 Factors once only

B3 Deduction root from factor

B4 Root formula once only

Attempts

A1 Not quadratic

Worthless

W1 Trial and error only

Part (b)	**20 (5, 5, 10) marks**	**Att (2, 2, 3)**

2 (b) (i) Find the range of values of $t \in \mathbf{R}$ for which the quadratic equation

$$(2t-1)x^2 + 5tx + 2t = 0 \text{ has real roots.}$$

(ii) Explain why the roots are real when t is an integer.

Correct substitution in $b^2 - 4ac \geq 0$	**5 marks**	**Att 2**
Inequality	**5 marks**	**Att 2**
Finish	**10 marks**	**Att 3**

2 (b) (i) $(2t-1)x^2 + 5tx + 2t = 0$

Real Roots: $b^2 - 4ac \geq 0$

$$(5t)^2 - 4(2t-1)(2t) \geq 0$$

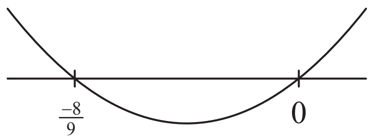
$$25t^2 - 16t^2 + 8t \geq 0$$

$$9t^2 + 8t \geq 0$$

Graph $9t^2 + 8t = 0$

$$t(9t+8) = 0$$

$t = 0$ **or** $t = -\frac{8}{9}$



$$\therefore\ 9t^2 + 8t \geq 0 \text{ when } \{t \leq -\tfrac{8}{9}\} \cup \{t \geq 0\}$$

(ii) Imaginary roots only when $-\frac{8}{9} < t < 0$

No integer included here.

$\Rightarrow$ real roots for all integers.

Blunders (-3)

B1 Inequality sign
B2 Indices
B3 Factors once only
B4 Deduction root from factor
B5 Range not stated (written down) or no range
B6 Incorrect range
B7 Shade graph only
B8 Incorrect deduction or no deduction in (ii)

Misreading (-1)

M1 Uses ‘>’ for '≥'

Part (c) **15 (5, 5, 5) marks** **Att (2, 2, 2)**

2 (c)	$f(x) = 1 - b^{2x}$ and $g(x) = b^{1+2x}$, where b is a positive real number. Find, in terms of b, the value of x for which $f(x) = g(x)$.

Equation	**5 marks**	**Att 2**
b^{2x} **isolated**	**5 marks**	**Att 2**
Value	**5 marks**	**Att 2**

2 (c) $f(x) = 1 - b^{2x}$ $\qquad g(x) = b^{1+2x}$

$$f(x) = g(x)$$
$$1 - b^{2x} = b^{1+2x}$$
$$1 - b^{2x} = b.b^{2x}$$
$$1 = b.b^{2x} + b^{2x}$$
$$1 = b^{2x}(b+1)$$
$$b^{2x} = \frac{1}{b+1}$$
$$\log_b\left(b^{2x}\right) = \log_b\left(\tfrac{1}{b+1}\right)$$
$$2x\log_b b = -\log_b(b+1)$$
$$2x = -\log_b(b+1)$$
$$x = -\tfrac{1}{2}\log_b(b+1)$$
$$x = -\log_b\sqrt{b+1}$$

* Accept logs to <u>any</u> base

Blunders (-3)

B1 Indices
B2 Factors
B3 Logs

QUESTION 3

Part (a)	**5 marks**	**Att 2**
Part (b)	**25 (10, 5, 5, 5) marks**	**Att (3, 2, 2, 2)**
Part (c)	**20 (5, 5, 5, 5) marks**	**Att (2, 2, 2, 2)**

Part (a) **5 marks** **Att 2**

3. **(a)** Given that $z = 2 + i$, where $i^2 = -1$, find the real number d such that $z + \frac{d}{z}$ is real.

Part (a) **5 marks** **Att 2**

3 (a) $(2+i) + \left(\frac{d}{2+i}\right)$

$\frac{d}{2+i} = d\left[\frac{1}{2+i} \cdot \frac{2-i}{2-i}\right] = \frac{d(2-i)}{5}$

$(2+i) + \frac{d}{5}(2-i) = a + (0)i$

$\left(2 + \frac{2d}{5}\right) + \left(1 - \frac{d}{5}\right)i = (a) + (0)i$

Equating Coefficients : $1 - \frac{d}{5} = 0$

$1 = \frac{d}{5}$

$d = 5$

Blunders *(-3)*

B1 i

B2 Not real to real etc

B3 $(2+i)(2-i) \neq 5$

B4 Conjugate

Attempts

A1 $f(z) = 0$

Part (b)(i) **15(10, 5) marks** **Att (3, 2)**

3 (b) **(i)** Use matrix methods to solve the simultaneous equations

$4x - 2y = 5$

$8x + 3y = -4$

Part (b)(i) Matrix form **10 marks** **Att 3**

Solution **5 marks** **Att 2**

3 (b) (i) Solve : $4x - 2y = 5$

$8x + 3y = -4$

$$\begin{pmatrix} 4 & -2 \\ 8 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 5 \\ -4 \end{pmatrix} \qquad A = \begin{pmatrix} 4 & -2 \\ 8 & 3 \end{pmatrix} \Rightarrow A^{-1} = \frac{1}{28}\begin{pmatrix} 3 & 2 \\ -8 & 4 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{28}\begin{pmatrix} 3 & 2 \\ -8 & 4 \end{pmatrix}\begin{pmatrix} 5 \\ -4 \end{pmatrix}$$

$$= \frac{1}{28}\begin{pmatrix} 7 \\ -56 \end{pmatrix} \quad = \begin{pmatrix} \frac{1}{4} \\ -2 \end{pmatrix}$$

Blunders (-3)
B1 Incorrect matrix A
B2 Incorrect $\frac{1}{\text{det}}$ or no $\frac{1}{\text{det}}$
B3 $A^{-1}.A \neq I$
B4 Incorrect matrix

Part (b) (ii) **10 (5, 5)marks** **Att (2, 2)**

3 (b) (ii) Find the two values of k which satisfy the matrix equation

$$\begin{pmatrix}1 & k\end{pmatrix}\begin{pmatrix}3 & 4\\ -2 & 1\end{pmatrix}\begin{pmatrix}1\\ k\end{pmatrix}=11.$$

Part (b)(ii) Complete Multiplication **5 marks** **Att 2**
Values **5 marks** **Att 2**

3 (b) (ii)

$$\begin{pmatrix}1 & k\end{pmatrix}\begin{pmatrix}3 & 4\\ -2 & 1\end{pmatrix}\begin{pmatrix}1\\ k\end{pmatrix}=11$$
$$\begin{pmatrix}1 & k\end{pmatrix}\begin{pmatrix}3+4k\\ -2+k\end{pmatrix}=11$$
$$3+4k-2k+k^2-11=0$$
$$k^2+2k-8=0$$
$$(k+4)(k-2)=0$$
$$k=-4 \quad \text{or} \quad k=2$$

or

$$\begin{pmatrix}1 & k\end{pmatrix}\begin{pmatrix}3 & 4\\ -2 & 1\end{pmatrix}\begin{pmatrix}1\\ k\end{pmatrix}=11$$
$$\begin{pmatrix}3-2k & 4+k\end{pmatrix}\begin{pmatrix}1\\ k\end{pmatrix}=11$$
$$3-2k+4k+k^2=11$$
$$k^2+2k-8=0$$
$$(k+4)(k-2)=0$$
$$k=-4 \quad \text{or} \quad k=2$$

Blunders (-3)
B1 Indices
B2 Factors once only
B3 Deduction root from factor or no deduction.
B4 Incorrect matrix

Note: Cannot get 2nd 5 marks if equation is linear.

Part (c)	**20 (5, 5, 5, 5)**	**Att (2, 2, 2, 2)**

3 (c) **(i)** Express $-8-8\sqrt{3}\,i$ in the form $r(\cos\theta + i\sin\theta)$.

(ii) Hence find $\left(-8-8\sqrt{3}\,i\right)^3$.

(iii) Find the four complex numbers z such that

$$z^4 = -8-8\sqrt{3}\,i.$$

Give your answers in the form $a+bi$, with a and b fully evaluated

(i)	**5 marks**	**Att 2**
(ii)	**5 marks**	**Att 2**
(iii) z	**5 marks**	**Att 2**
Complex numbers	**5 marks**	**Att 2**

3(c)(i)

$$z = -8-8\sqrt{3}i$$

$$r = \sqrt{(-8)^2 + \left(-8\sqrt{3}\right)^2}$$

$$= \sqrt{64+192}$$

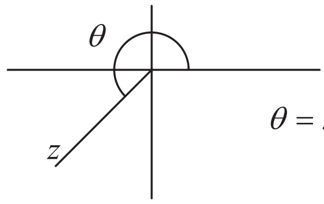
$$= \sqrt{256}$$

$$=16$$

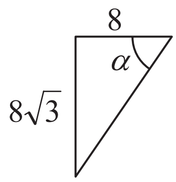
$$z = r[\cos\theta + i\sin\theta]$$

$$= 16\left[\cos\tfrac{4\pi}{3} + i\sin\tfrac{4\pi}{3}\right]$$

$$= 2^4\left(\cos\tfrac{4\pi}{3} + i\sin\tfrac{4\pi}{3}\right)$$



$\theta = \pi + \alpha$



$\tan\alpha = \frac{8\sqrt{3}}{8} = \sqrt{3}$

$\alpha = \frac{\pi}{3} \quad \Rightarrow \quad \theta = \frac{4\pi}{3}$

3(c)(ii)

$$z = 2^4\left(\cos\tfrac{4\pi}{3} + i\sin\tfrac{4\pi}{3}\right)$$

$$z^3 = \left[2^4\left(\cos\tfrac{4\pi}{3} + i\sin\tfrac{4\pi}{3}\right)\right]^3$$

$$= 2^{12}\left(\cos 4\pi + i\sin 4\pi\right)$$

$$= 2^{12}\left(1 + i(0)\right)$$

$$= 2^{12} = 4096$$

3(c)(iii)

$$z^4 = -8-8\sqrt{3}i = 2^4\left[\cos\left(2n\pi + \tfrac{4\pi}{3}\right) + i\sin\left(2n\pi + \tfrac{4\pi}{3}\right)\right]$$

$$\Rightarrow z = \left\langle 2^4\left[\cos\left(2n\pi + \tfrac{4\pi}{3}\right) + i\sin\left(2n\pi + \tfrac{4\pi}{3}\right)\right]\right\rangle^{\frac{1}{4}}$$

$$z = 2\left[\cos\left(\tfrac{n\pi}{2} + \tfrac{\pi}{3}\right) + i\sin\left(\tfrac{n\pi}{2} + \tfrac{\pi}{3}\right)\right]$$

$n=0: z_0 = 2\left[\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right] \quad = 2\left[\frac{1}{2} + i\frac{\sqrt{3}}{2}\right] \quad = 1 + i\sqrt{3}$

$n=1: z_1 = 2\left[\cos\left(\frac{\pi}{2} + \frac{\pi}{3}\right) + i\sin\left(\frac{\pi}{2} + \frac{\pi}{3}\right)\right]$

$= 2\left[\cos\left(\frac{5\pi}{6}\right) + i\sin\left(\frac{5\pi}{6}\right)\right] \quad = 2\left[-\frac{\sqrt{3}}{2} + i\left(\frac{1}{2}\right)\right] \quad = -\sqrt{3} + i$

$n=2: z_2 = 2\left[\cos\left(\pi + \frac{\pi}{3}\right) + i\sin\left(\pi + \frac{\pi}{3}\right)\right]$

$= 2\left[\cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3}\right] \quad = 2\left[-\frac{1}{2} - i\left(\frac{\sqrt{3}}{2}\right)\right] \quad = -1 - i\sqrt{3}$

$n=3: z_3 = 2\left[\cos\left(\frac{3\pi}{2} + \frac{\pi}{3}\right) + i\sin\left(\frac{3\pi}{2} + \frac{\pi}{3}\right)\right]$

$= 2\left[\cos\frac{11\pi}{6} + i\sin\frac{11\pi}{6}\right] \quad = 2\left[\frac{\sqrt{3}}{2} + i\left(-\frac{1}{2}\right)\right] \quad = \sqrt{3} - i$

Blunders (-3)
B1 Argument
B2 Modulus
B3 Trig definition
B4 Indices
B5 *i*
B6 Statement De Moivre once only
B7 Application De Moivre
B8 Root omitted
B9 No general solution
B10 *a* and *b* not fully evaluated
B11 Improper use of polar formula

Slips (-1)
S1 Trig value

QUESTION 4

Part (a)	**10 (5, 5) marks**	**Att (2, 2)**
Part (b)	**20 (5, 5, 5, 5) marks**	**Att (2, 2, 2, 2)**
Part (c)	**20 (5, 5, 5, 5) marks**	**Att (2, 2, 2, 2)**

Part (a)	**10 (5, 5) marks**	**Att (2, 2)**

4 (a) $-2+2+6+...+(4n-6)$ are the first n terms of an arithmetic series.
S_n, the sum of these n terms, is 160. Find the value of n.

Correct substitution in formula	**5 marks**	**Att 2**
Value n	**5 marks**	**Att 2**

4 (a) $-2+2+6+...+(4n-6)$

$$a=-2 \quad ; \quad d=4 \quad ; \quad S_n=160$$

$$S_n=\tfrac{n}{2}\left[2a+(n-1)d\right]=160$$

$$\tfrac{n}{2}\left[-4+(n-1)4\right]=160$$

$$\tfrac{n}{2}\left[-4+4n-4\right]=160$$

$$n(2n-4)=160$$

$$2n^2-4n-160=0$$

$$n^2-2n-80=0$$

$$(n-10)(n+8)=0$$

$$n=10 \quad \text{or} \quad n=-8$$

$$n=10$$

Blunders (-3)
B1 Formula *AP* once only
B2 Incorrect *'a'* in formula
B3 Incorrect *'d'* in formula
B4 Indices
B5 Factors once only
B6 Incorrect deduction root from factor
B7 Roots formula

Slips
S1 Excess value or incorrect value indicated.

Worthless
W1 treats as *G.P.*

Part (b)	20 (5, 5, 5, 5) marks	Att (2, 2, 2, 2)

4 (b) The sum to infinity of a geometric series is $\frac{9}{2}$.

The second term of the series is -2.
Find the value of r, the common ratio of the series.

Part (b)			
	1^{st} equation	5 marks	Att 2
	2^{nd} equation	5 marks	Att 2
	Quadratic simplified	5 marks	Att 2
	Value r	5 marks	Att 2

4 (b)

$$S_\infty = \frac{9}{2} = \frac{a}{1-r}$$
$$9(1-r) = 2a \ldots\ldots\ldots\ldots\ldots\ldots\ldots (i)$$

$$a, ar, ar^2$$
$$ar = -2$$
$$a = -\frac{2}{r} \ldots\ldots\ldots\ldots\ldots\ldots\ldots (ii)$$

$$(i): 9(1-r) = 2(a)$$
$$9(1-r) = 2\left(-\frac{2}{r}\right)$$
$$9 - 9r = -\frac{4}{r}$$
$$9r - 9r^2 = -4$$
$$0 = 9r^2 - 9r - 4$$
$$0 = (3r+1)(3r-4)$$
$$\Rightarrow r = -\frac{1}{3} \quad \text{or} \quad r = \frac{4}{3}$$

Since sum to infinity $\Rightarrow r = -\frac{1}{3}$

Blunders (-3)
B1 Formula sum to infinity
B2 Definition of term of a *G.P.*
B3 Indices
B4 Factors once only
B5 Incorrect deduction root from factor
B6 Incorrect '*a*'

Slips
S1 Excess value or incorrect value indicated

Worthless
W1 Uses *AP* formula
W2 Trial and error

Part (c)	20 (5, 5, 5, 5) marks	Att(2, 2, 2, 2)

4 (c) The sequence $u_1, u_2, u_3, \ldots$, defined by $u_1 = 3$ and $u_{n+1} = 2u_n + 3$, is as follows:
3, 9, 21, 45, 93……

(i) Find u_6**,** and verify that it is equal to the sum of the first six terms of a geometric series with first term 3 and common ratio 2.

(ii) Given that, for all *k*, u_k is the sum of the first *k* terms of a geometric series with first term 3 and common ratio 2, find $\sum_{k=1}^{n} u_k$.

Part (c)(i)	U_6	**5 marks**	**Att 2**
	Verify	**5 marks**	**Att 2**
Part (c)(ii)	**Showing Pattern**	**5 marks**	**Att 2**
	Sum	**5 marks**	**Att 2**

4 (c) (i) $u_1 = 3 \quad ; \quad u_{n+1} = 2u_n + 3$

$$u_6 = 2(u_5) + 3 \qquad = 2(93) + 3 \qquad = 189$$

$$G.P.: \quad a = 3 \;;\quad r = 2$$

$$s_6 = \frac{3[2^6 - 1]}{2 - 1} = 3[64 - 1] = 189$$

or

$$s_6 = 3 + 6 + 12 + 24 + 48 + 96 = 189$$

Q4(c)(ii) (3), (3+6), (3+6+12), (3+6+12+24),…
(3), (9), (21), (45)

$$u_k = 3(2^k - 1) = 3(2^k) - 3$$

$$u_n = 3.2^n - 3$$
$$u_{n-1} = 3.2^{n-1} - 3$$
$$u_{n-2} = 3.2^{n-2} - 3$$
…………………..
$$u_3 = 3.2^3 - 3$$
$$u_2 = 3.2^2 - 3$$
$$u_1 = 3.2^1 - 3$$

$$\sum = 3(2^1 + 2^2 + \ldots\ldots + 2^n) - 3n$$
$$= 3\left[\frac{2(2^n - 1)}{2 - 1}\right] - 3n = 6(2^n - 1) - 3n$$

Blunders (-3)

B1 Error in U_6
B2 Formula sum of *G.P.*
B3 Incorrect *'a'*
B4 Incorrect *'R'*
B5 Error in U_k
B6 Indices

QUESTION 5

Part (a)	**10 marks**	**Att 3**
Part (b)	**20 (5, 10, 5) marks**	**Att (2, 3, 2)**
Part (c)	**20 (5, 5, 5, 5) marks**	**Att (2, 2, 2, 2)**

Part (a)	**10 marks**	**Att 3**

5 (a) Find the value of the middle term of the binomial expansion of

$$\left(\frac{x}{y}-\frac{y}{x}\right)^8.$$

Part (a)	**10 marks**	**Att 3**

5 (a) $u_5=\binom{8}{4}\left[\left(\frac{x}{y}\right)^4\left(-\frac{y}{x}\right)^4\right] \qquad =\frac{8.7.6.5}{1.2.3.4}(1)=70$

or

5(a)

$$\left(\frac{x}{y}-\frac{y}{x}\right)^8=\left(\frac{x}{y}\right)^8+\binom{8}{1}\left(\frac{x}{y}\right)^7\left(-\frac{y}{x}\right)+\binom{8}{2}\left(\frac{x}{y}\right)^6\left(-\frac{y}{x}\right)^2+\binom{8}{3}\left(\frac{x}{y}\right)^5\left(-\frac{y}{x}\right)^3+\binom{8}{4}\left(\frac{x}{y}\right)^4\left(-\frac{y}{4}\right)^4+\text{.........}$$

$$u_5=\binom{8}{4}\left(\frac{x}{y}\right)^4\left(-\frac{y}{x}\right)^4=70$$

* Answer must be in simplest form

Blunders (-3)

B1 General term
B2 Errors in binominal expansion once only
B3 Indices
B4 error value $\binom{n}{r}$ or no value $\binom{n}{r}$
B5 x and y in answer
B6 $x^\circ \neq 1$
B7 Incorrect term

Part (b)	**20 (5, 10, 5) marks**	**Att (2, 3, 2)**

5 (b) **(i)** Express $\frac{2}{(r+1)(r+3)}$ in the form $\frac{A}{r+1}+\frac{B}{r+3}$.

(ii) Hence find $\sum_{r=1}^{n}\frac{2}{(r+1)(r+3)}$.

(iii) Hence evaluate $\sum_{r=1}^{\infty}\frac{2}{(r+1)(r+3)}$.

Part (b) (i)	**5 marks**	**Att 2**
(ii)	**10 marks**	**Att 3**
(iii)	**5 marks**	**Att 2**

5 (b) (i)

$$\frac{a}{r+1}+\frac{b}{r+3}=\frac{2}{(r+1)(r+3)}$$

$$a(r+3)+b(r+1)=2$$

$$ar+3a+br+b=(0)r+(2)$$

$$(a+b)r+(3a+b)=(0)r+2$$

Equating Coefficients: (i) $a+b=0 \Rightarrow a=-b$

(ii) $3a+b=2$

$3a-a=2$

$2a=2 \Rightarrow a=1$ and $b=-1$

$$\frac{2}{(r+1)(r+3)}=\frac{1}{r+1}-\frac{1}{r+3}$$

Q5(b)(ii)

$$\sum_{r=1}^{n}\frac{2}{(r+1)(r+3)}$$

$$u_n=\frac{2}{(n+1)(n+3)} \qquad =\frac{1}{n+1}-\frac{1}{n+3}$$

$$u_{n-1}=\frac{2}{n(n+2)} \qquad =\frac{1}{n}-\frac{1}{n+2}$$

$$u_{n-2}=\frac{2}{(n-1)(n+1)} \qquad =\frac{1}{n-1}-\frac{1}{n+1}$$

$$\vdots \qquad \vdots \qquad \vdots$$

$$u_3=\frac{2}{4.6} \qquad =\frac{1}{4}-\frac{1}{6}$$

$$u_2=\frac{2}{3.5} \qquad =\frac{1}{3}-\frac{1}{5}$$

$$u_1=\frac{2}{2.4} \qquad =\frac{1}{2}-\frac{1}{4}$$

$$s_n=\frac{1}{2}+\frac{1}{3}-\frac{1}{n+2}-\frac{1}{n+3} \qquad =\frac{5}{6}-\frac{1}{n+2}-\frac{1}{n+3}$$

Q5(b)(iii) $n \to \infty$ $\qquad s_\infty=\frac{5}{6}$

Blunders (-3)

B1 Indices

B2 Cancellation must be shown or implied

B3 In equating coefficients not like to like

B4 Term or terms omitted

B5 S_r

B6 limits

Note: Must show 3 terms at start and 2 terms at finish or vice versa otherwise attempt.

Part (c)	20 (5, 5, 5, 5) marks	Att (2, 2, 2, 2)

5 (c) (i) Given two real numbers a and b, where $a>1$ and $b>1$, prove that

$$\frac{1}{log_b a}+\frac{1}{log_a b}\geq 2.$$

(ii) Under what condition is $\frac{1}{\log_b a}+\frac{1}{\log_a b}=2$.

Part 5(c) (i) Change of Base	5 marks	Att 2
Inequality	5 marks	Att 2
Prove	5 marks	Att 2
(ii)	5 marks	Att 2

5 (c) (i) $\frac{1}{\log_b a}+\frac{1}{\log_a b}\geq 2$

$\Leftrightarrow \log_a b+\frac{1}{\log_a b}\geq 2$ (since $\log_a b=\frac{1}{\log_b a}$)

$\Leftrightarrow (\log_a b)^2+1\geq 2\log_a b$ (since $a>1, b>1 \Rightarrow \log_a b>0$)

$\Leftrightarrow (\log_a b)^2+1-2\log_a b\geq 0$

$\Leftrightarrow (\log_a b)^2-2(\log_a b)+1\geq 0$

$\Leftrightarrow (\log_a b-1)^2\geq 0$

True, so $\frac{1}{\log_b a}+\frac{1}{\log_a b}\geq 2.$

or

Q5 (c) (i) To prove: $\log_a b+\frac{1}{\log_a b}\geq 2$

Let $\log_a b=x$. Then $x>0$, since $a>1, b>1 \Rightarrow \log_a b>0$.

To prove: $x+\frac{1}{x}\geq 2$

$\Leftrightarrow x^2+1\geq 2x$

$\Leftrightarrow x^2-2x+1\geq 0$

$\Leftrightarrow (x-1)^2\geq 0$

True, so $x+\frac{1}{x}\geq 2$. That is, $\log_a b+\frac{1}{\log_a b}\geq 2$

Q5(c)(ii) Equality holds in above solution when $(\log_a b-1)^2=0$ [or $(x-1)^2=0$ in 2nd version]

$\Leftrightarrow \log_a b=1$

$\Leftrightarrow a=b$

Blunders (-3)

B1 Log laws
B2 Change of base
B3 Inequality sign
B4 Incorrect deduction or no deduction
B5 Indices
B6 Factors once only
B7 $(x^2-2x+1)\neq(x-1)^2$

Note : Inequality must be quadratic

QUESTION 6

Part (a)	**15 marks**	**Att 5**
Part (b)	**20 (10, 5, 5) marks**	**Att (3, 2, 2)**
Part (c)	**15 (5, 5, 5) marks**	**Att (2, 2, 2)**

Part (a) **15 marks** **Att 5**

6 (a) Differentiate $\sqrt{x}(x+2)$ with respect to x.

Part (a) **15 marks** **Att 5**

6 (a)

$$y = \sqrt{x}.(x+2)$$
$$= x^{\frac{1}{2}}(x+2)$$
$$= x^{\frac{3}{2}} + 2x^{\frac{1}{2}}$$
$$\frac{dy}{dx} = \frac{3}{2}\left(x^{\frac{1}{2}}\right) + 2\left(\frac{1}{2}.x^{-\frac{1}{2}}\right)$$
$$= \frac{3}{2}\left(x^{\frac{1}{2}}\right) + \frac{1}{x^{\frac{1}{2}}}$$

or

6(a)

$$y = x^{\frac{1}{2}}.(x+2)$$
$$\frac{dy}{dx} = x^{\frac{1}{2}}(1) + (x+2)\left(\tfrac{1}{2}.x^{-\frac{1}{2}}\right)$$
$$= x^{\frac{1}{2}} + \frac{x+2}{2x^{\frac{1}{2}}}$$
$$= \sqrt{x} + \frac{x+2}{2\sqrt{x}}$$

Blunders (-3)
B1 Indices
B2 Differentiation

Attempts
A1 Error in differentiation formula

Part (b)	20 (10, 5, 5) marks	Att (3, 2, 2)

6 (b) The equation of a curve is $y = 3x^4 - 2x^3 - 9x^2 + 8$.

(i) Show that the curve has a local maximum at the point (0, 8).

(ii) Find the coordinates of the two local minimum points on the curve.

(iii) Draw a sketch of the curve.

Part (b) (i)	10 marks	Att 3
(ii)	5 marks	Att 2
(iii)	5 marks	Att 2

6 (b) (i) $y = 3x^4 - 2x^3 - 9x^2 + 8$.

$$\frac{dy}{dx} = 12x^3 - 6x^2 - 18x$$

$$\frac{d^2y}{dx^2} = 36x^2 - 12x - 18$$

Local Max/Min: $\frac{dy}{dx} = 0 \Rightarrow 12x^3 - 6x^2 - 18x = 0$

$$6x(2x^2 - x - 3) = 0$$

$$x = 0 \quad \text{or} \quad 2x^2 - x - 3 = 0$$

$$(2x-3)(x+1) = 0$$

$$x = \tfrac{3}{2} \quad \text{or} \quad x = -1$$

Test $x = 0$ in $\frac{d^2y}{dx^2}$ $= 36(0) - 12(0) - 18 \quad = -18 \quad < 0 \quad \Rightarrow$ local max at $x = 0$.

When $x = 0$: $y = 3(0) - 2(0) - 9(0) + 8 \quad = 8 \qquad \therefore$ Local max at $(0,8)$

Q6(b)(ii) Other two turning points at $x = \frac{3}{2}$ and $x = -1$

At $x = \frac{3}{2}$: $y = 3\left(\tfrac{3}{2}\right)^4 - 2\left(\tfrac{3}{2}\right)^3 - 9\left(\tfrac{3}{2}\right)^2 + 8$

$$= \frac{243}{16} - \frac{108}{16} - \frac{324}{16} + \frac{128}{16}$$

$$= -\frac{61}{16} = -3\frac{13}{16} \approx -3.8 \qquad \Rightarrow y \approx -3.8$$

And at $x = \frac{3}{2}$: $\frac{d^2y}{dx^2} = 36\left(\tfrac{9}{4}\right) - 12\left(\tfrac{3}{2}\right) - 18 \quad = 81 - 18 - 18 > 0 \quad \Rightarrow$ local min at $\left(\tfrac{3}{2}, -3.8\right)$

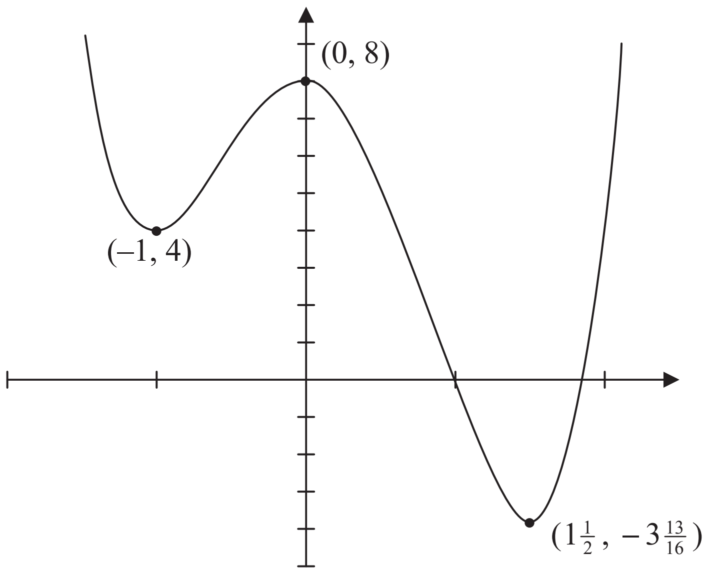
At $x = -1$: $y = 3(-1)^4 - 2(-1)^3 - 9(-1)^2 + 8$

$$= 3 + 2 - 9 + 8$$

$$= 4 \qquad \Rightarrow y = 4$$

And at $x = -1$: $\frac{d^2y}{dx^2} = 36(-1)^2 - 12(-1) - 18 \quad = 36 + 12 - 18 > 0 \quad \Rightarrow$ local min at (–1, 4).

Q6(b)(iii)



Blunders (-3)

B1 Differentiation
B2 Indices
B3 Deduction from 2nd derivative or no deduction
B4 Not 3 values from $f'(x) = 0$
B5 Not testing in $f''(x)$ for max
B6 Incorrect y values or no y values in (ii)
B7 Factors once only
B8 Incorrect root from factor
B9 Not getting $f''(x)$

Attempts

A1 Error in differentiation formula

Worthless

W1 Integration

Part (c) **15 (5, 5, 5) marks** **Att (2, 2, 2)**

6 (c)	Prove by induction that $\frac{d}{dx}\left(x^n\right) = nx^{n-1},\ n \ge 1,\ n \in \mathbf{N}$.

6(c) P(1)	**5 marks**	**Att 2**
P(k)	**5 marks**	**Att 2**
P(k +1)	**5 marks**	**Att 2**

6 (c) $P(n) : \frac{d}{dx}\left(x^n\right) = nx^{n-1}$

$P(1)$: show that $\frac{d}{dx}(x) = 1$:

$$f(x) = x \Rightarrow \frac{f(x+h) - f(x)}{h} = \frac{(x+h) - x}{h} = \frac{h}{h} = 1.$$

$$\therefore \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = 1 \qquad \therefore P(1) \text{ is true.}$$

Assume $P(k)$ true: $\frac{d}{dx}\left(x^k\right) = kx^{k-1}$

Deduce $P(k+1)$: $\frac{d}{dx}\left(x^{k+1}\right) = \frac{d}{dx}\left(x^k . x\right)$

$= x^k(1) + x\frac{d}{dx}\left(x^k\right)$ (by product rule)

$= x^k(1) + x\left(kx^{k-1}\right)$ (by $P(k)$ assumption)

$= x^k + kx^k$

$= x^k(k+1)$

$\therefore$ True for $p(k+1)$

As $P(1)$ is true, and $P(k) \Rightarrow P(k+1)$ for all k, $P(n)$ is true for all $n \ge 1$.

Blunders (-3)

B1 Failure to prove case *n = 1* or uses rule to prove true for *n = 1*

B2 Definition of $f'(x)$

B3 Error in $f(x+k)$ or $(x + \Delta x)$

B4 Indices

B5 Limit or no limit shown or implied

B6 Differentiation

B7 *n = 0*

Attempts

A1 Error in differentiation formula

QUESTION 7

Part (a)	**15 (5, 5, 5) marks**	**Att (2, 2, 2)**
Part (b)	**20 (5, 5, 5, 5) marks**	**Att (2, 2, 2, 2)**
Part (c)	**15 (5, 5, 5) marks**	**Att (2, 2, 2)**

Part (a)	**15 (5, 5, 5) marks**	**Att (2, 2, 2)**

7 (a) Taking $x_1 = 2$ as the first approximation to the real root of the equation

$$x^3 + x - 9 = 0,$$

use the Newton-Raphson method to find x_2, the second approximation.

Part (a) Formula	**5 marks**	**Att 2**
Differentiation	**5 marks**	**Att 2**
Finish	**5 marks**	**Att 2**

7 (a)

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$x_1 = 2: \quad f(x) = x^3 + x - 9 \quad \Rightarrow f(2) = (2)^3 + 2 - 9 = 1$$

$$f'(x) = 3x^2 + 1 \quad \Rightarrow f'(2) = 3(2)^2 + 1 = 13$$

$$x_2 = 2 - \tfrac{1}{13} = \tfrac{25}{13}$$

Blunder (-3)

B1 Newton-Raphson formula once only

B2 Differentiation

B3 Indices

B4 $x_1 \neq 2$

Worthless

W1 Incorrect answer and no work

Part (b)	**20 (5, 5, 5, 5) marks**	**Att (2, 2, 2, 2)**

7 (b) The parametric equations of a curve are:

$$x = 3\cos\theta - \cos^3\theta$$

$$y = 3\sin\theta - \sin^3\theta, \text{ where } 0 < \theta < \frac{\pi}{2}.$$

(i) Find $\frac{dy}{d\theta}$ and $\frac{dx}{d\theta}$.

(ii) Hence show that $\frac{dy}{dx} = \frac{-1}{\tan^3\theta}$.

$\frac{dx}{d\theta}$	**5 marks**	**Att 2**
$\frac{dy}{d\theta}$	**5 marks**	**Att 2**
$\frac{dy}{dx}$	**5 marks**	**Att 2**
Show	**5 marks**	**Att 2**

7 (b)

$$x = 3\cos\theta - (\cos\theta)^3$$

$$\frac{dx}{d\theta} = -3\sin\theta - 3(\cos\theta)^2.(-\sin\theta)$$

$$= -3\sin\theta + 3\sin\theta\cos^2\theta$$

$$= -3\sin\theta(1 - \cos^2\theta)$$

$$= -3\sin^3\theta$$

$$y = 3\sin\theta - (\sin\theta)^3$$

$$\frac{dy}{d\theta} = 3\cos\theta - 3(\sin\theta)^2.\cos\theta$$

$$= 3\cos\theta(1 - \sin^2\theta)$$

$$= 3\cos^3\theta$$

$$\frac{dy}{dx} = \frac{\left(\frac{dy}{d\theta}\right)}{\left(\frac{dx}{d\theta}\right)} = \frac{3\cos^3\theta}{-3\sin^3\theta} = -\frac{1}{\left(\frac{\sin\theta}{\cos\theta}\right)^3} \quad = -\frac{1}{\tan^3\theta}$$

Blunders (-3)

B1 Indices
B2 Differentiation
B3 Trig laws
B4 Not in required form

Attempts

A1 Error in differentiation formula

Part (c)	**15 (5, 5, 5) marks**	**Att (2, 2, 2)**

7 (c) Given $y = \ln\left(\dfrac{3+x}{\sqrt{9-x^2}}\right)$, find $\dfrac{dy}{dx}$ and express it in the form $\dfrac{a}{b-x^n}$.

$\dfrac{dy}{dx}$	**5 marks**	**Att 2**
Simplifying	**5 marks**	**Att 2**
Express	**5 marks**	**Att 2**

7 (c)

$$y = \ln\left(\frac{3+x}{\sqrt{9-x^2}}\right)$$

$$= \ln(3+x) - \ln\sqrt{9-x^2}$$

$$= \ln(3+x) - \tfrac{1}{2}\ln(9-x^2)$$

$$\frac{dy}{dx} = \frac{1}{3+x} - \frac{1}{2}\left[\frac{1}{9-x^2}(-2x)\right]$$

$$= \frac{1}{3+x} + \frac{x}{9-x^2}$$

$$= \frac{1}{3+x} + \frac{x}{(3-x)(3+x)}$$

$$= \frac{(3-x)+x}{(3-x)(3+x)} = \frac{3}{9-x^2}$$

or

Q7(c)

$$y = \ln\frac{3+x}{\sqrt{(3-x).(3+x)}}$$

$$= \ln\frac{(3+x)^{\frac{1}{2}}}{(3-x)^{\frac{1}{2}}}$$

$$= \tfrac{1}{2}\ln(3+x) - \tfrac{1}{2}\ln(3-x)$$

$$\frac{dy}{dx} = \frac{1}{2}\left[\frac{1}{3+x} - \frac{1}{3-x}(-1)\right]$$

$$= \frac{1}{2}\left[\frac{1}{3+x} + \frac{1}{3-x}\right]$$

$$= \frac{1}{2}\left[\frac{(3-x)+(3+x)}{9-x^2}\right] = \frac{1}{2}\left(\frac{6}{9-x^2}\right) = \frac{3}{9-x^2}$$

or

Q7(c)

$$y = \ln\left(\frac{3+x}{\sqrt{9-x^2}}\right) = \ln\left(\frac{3+x}{\left(9-x^2\right)^{\frac{1}{2}}}\right)$$

$$\frac{dy}{dx} = \frac{1}{\left[\frac{3+x}{\left(9-x^2\right)^{\frac{1}{2}}}\right]} \cdot \frac{\left(9-x^2\right)^{\frac{1}{2}}(1) - (3+x)\frac{1}{2}\left(9-x^2\right)^{-\frac{1}{2}}.(-2x)}{\left(9-x^2\right)^1}$$

$$= \frac{\left(9-x^2\right)^{\frac{1}{2}}}{(3+x)} \cdot \frac{\left(9-x^2\right)^{\frac{1}{2}} + \frac{x(3+x)}{(9-x^2)^{\frac{1}{2}}}}{\left(9-x^2\right)}$$

$$= \frac{\left(9-x^2\right) + x(3+x)}{(3+x).\left(9-x^2\right)}$$

$$= \frac{9-x^2+3x+x^2}{(3+x)\left(9-x^2\right)}$$

$$= \frac{9+3x}{(3+x)\left(9-x^2\right)}$$

$$= \frac{3(3+x)}{(3+x)\left(9-x^2\right)}$$

$$= \frac{3}{9-x^2}$$

* $\frac{dy}{dx}$ and simplifying can be in any order

Blunders (-3)

B1 Differentiation
B2 Log laws
B3 Indices
B4 Not simplified to required form
B5 Factors once only.

Attempts

A1 Error in differentiation formula

QUESTION 8

Part (a)	**10 (5, 5) marks**	**Att (2, 2)**
Part (b)	**20 (5, 5, 5, 5) marks**	**Att (2, 2, 2, 2)**
Part (c)	**20 (10, 10) marks**	**Att (3, 3)**

Part (a) **10 (5, 5) marks** **Att (2, 2)**

8. **(a)** Find **(i)** $\int \sqrt{x}dx$ **(ii)** $\int e^{-2x}\,dx.$

Part (a) **10 (5, 5) marks** **Att (2, 2)**

Q8 (a)(i) $\int \sqrt{x}dx = \int x^{\frac{1}{2}}.dx = \dfrac{x^{\frac{3}{2}}}{\frac{3}{2}} + c \quad = \dfrac{2}{3}x^{\frac{3}{2}} + c$

Q8 (a)(ii) $\int e^{-2x}\,dx = -\tfrac{1}{2}e^{-2x} + c$

***** If c shown once, then no penalty

Blunders (-3)
B1 Integration
B2 Indices
B3 No *'c'* (penalize 1st integration)

Attempts
A1 Only *'c'* correct

Worthless
W1 Differentiation instead of integration

Part (b) **20 (5, 5, 5, 5) marks** **Att (2, 2, 2, 2)**

8 (b) Evaluate (i) $\int_1^2 x\left(1+x^2\right)^3 dx$ (ii) $\int_0^{\frac{\pi}{4}} \sin 5\theta \cos 3\theta \, d\theta.$

Part (b)(i)	**Integration**	**5 marks**	**Att 2**
	Value	**5 marks**	**Att 2**
Part (b)(ii)	**Integration**	**5 marks**	**Att 2**
	Value	**5 marks**	**Att 2**

8 (b) (i) $\int_1^2 x\left(1+x^2\right)^3 dx$

Let $u = 1+x^2$

$\frac{du}{dx} = 2x$

$\frac{du}{2} = xdx$

$\int \left(1+x^2\right)^3 .xdx \quad = \int u^3 \, \frac{du}{2} \quad = \dfrac{1}{2}\left[\dfrac{u^4}{4}\right]$

$= \frac{1}{8}\left[\left(1+x^2\right)^4\right]_1^2$

$= \frac{1}{8}\left[(5)^4 - (2)^4\right] \quad = \dfrac{609}{8}$

or

Q8(b)(i)	$$\int_1^2 x\left(1+x^2\right)^3 dx$$ $$= \int x\left(1+3x^2+3x^4+x^6\right)dx$$ $$= \int\left(x+3x^3+3x^5+x^7\right)dx$$ $$= \frac{x^2}{2}+\frac{3x^4}{4}+\frac{x^6}{2}+\frac{x^8}{8}\Big]_1^2$$ $$= \left(2+12+32+32\right)-\left(\tfrac{1}{2}+\tfrac{3}{4}+\tfrac{1}{2}+\tfrac{1}{8}\right)$$ $$= 78-\tfrac{17}{8} = 76\tfrac{1}{8} = \tfrac{609}{8}$$	$$\left(1+x^2\right)^3 = (1)^3+\binom{3}{1}(1)^2\left(x^2\right)+\binom{3}{2}(1)\left(x^2\right)^2+\left(x^2\right)^3$$ $$= 1+3x^2+3x^4+x^6$$

Q8(b)(ii)	$$\int_0^{\frac{\pi}{4}} \sin 5\theta \cos 3\theta \, d\theta.$$ $$= \frac{1}{2}\int 2\sin 5\theta\cos 3\theta d\theta$$ $$= \frac{1}{2}\int\left(\sin 8\theta+\sin 2\theta\right)d\theta$$ $$= \frac{1}{2}\left[-\frac{\cos 8\theta}{8}-\frac{\cos 2\theta}{2}\right]_0^{\frac{\pi}{4}}$$ $$= \tfrac{1}{2}\left[\left(-\tfrac{\cos 2\pi}{8}-\tfrac{\cos\frac{\pi}{2}}{2}\right)-\left(-\tfrac{\cos 0}{8}-\tfrac{\cos 0}{2}\right)\right]$$ $$= \tfrac{1}{2}\left[\left(-\tfrac{1}{8}-0\right)-\left(-\tfrac{1}{8}-\tfrac{1}{2}\right)\right]$$ $$= \tfrac{1}{2}\left[-\tfrac{1}{8}+\tfrac{1}{8}+\tfrac{1}{2}\right]$$ $$= \tfrac{1}{4}$$

Blunders (-3)
B1 Integration
B2 Indices
B3 Differentiation
B4 Limits
B5 Incorrect order in applying limits
B6 Not calculating substituted limits
B7 Not changing limits
B8 Trig formula

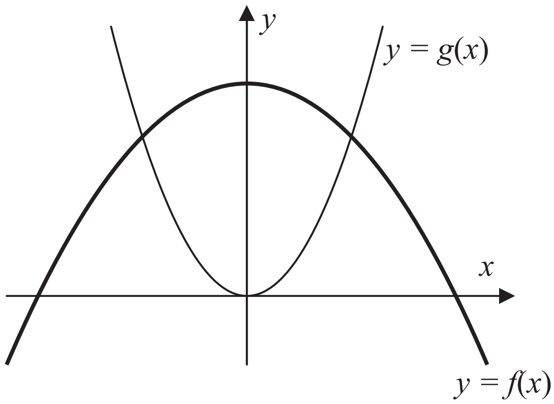
Slips
S1 Trig value

Worthless
W1 Differentiation instead of integration except where other work merits attempt

Note Incorrect substitution and unable to finish yields attempt at most.

Part (c) **20 (10, 10) marks** **Att (3, 3)**

8 (c) The diagram shows the graphs of the curves $y = f(x)$ and $y = g(x)$, where $f(x) = 12 - 3x^2$ and $g(x) = 9x^2$.



(i) Calculate the area of the region enclosed by the curve $y = f(x)$ the x-axis.

(ii) Show that the region enclosed by the curves $y = f(x)$ and $y = g(x)$ has half that area.

Part (c) (i) **10 marks** **Att 3**
(ii) **10 marks** **Att 3**

8 (c) (i)

$$f(x) = 0 \Rightarrow 12 - 3x^2 = 0$$

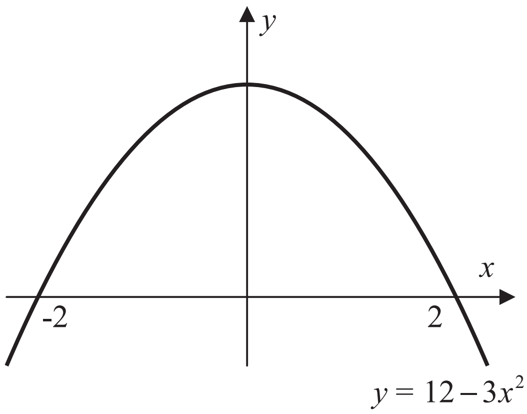
$$4 = x^2 \quad \Rightarrow \quad x = \pm 2$$

$$A = \int_{-2}^{2} y.dx = 2\int_{0}^{2} (12 - 3x^2)dx$$

$$= 2\left[12x - x^3\right]_0^2$$

$$= 2\left[(24 - 8) - 0\right]$$

$$= 32$$



Q8(c)(ii)

$$f(x) = g(x)$$

$$12 - 3x^2 = 9x^2$$

$$12 = 12x^2$$

$$1 = x^2 \Rightarrow x = \pm 1$$

Enclosed Area $= \int_{-1}^{1} f(x)dx - \int_{-1}^{1} g(x)dx$

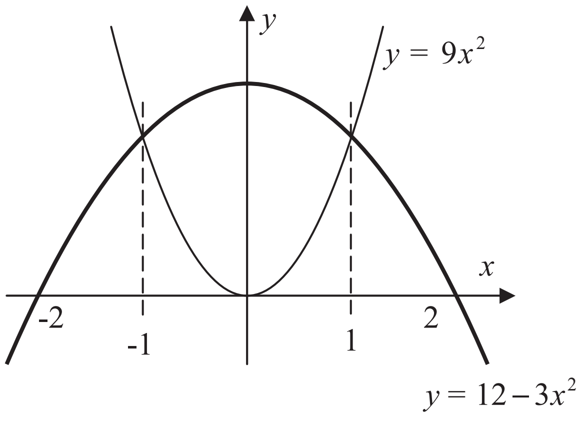
$$= 2\left[\int_0^1 (12 - 3x^2)dx - \int_0^1 9x^2 dx\right]$$

$$= 2\left[12x - x^3 - 3x^3\right]_0^1$$

$$= 2\left[12x - 4x^3\right]_0^1$$

$$= 2\left[(12 - 4) - (0)\right]$$

$$= 16$$



or

Q8(c)(ii) Enclosed Area $= 32 - [A_1 + A_2 + A_3 + A_4]$

$= 32 - 2(A_3 + A_4)$

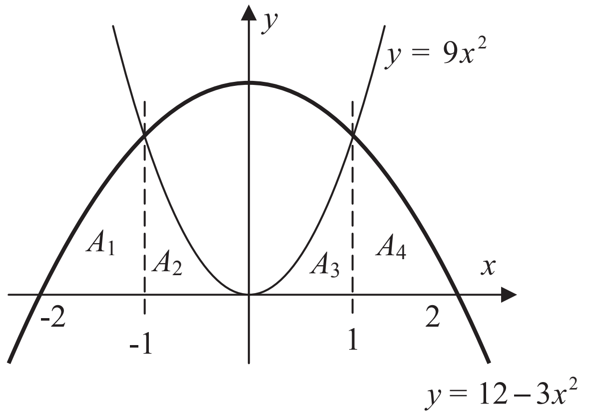
$$A_3 = \int_0^1 9x^2\,dx = 3x^3\Big]_0^1 = 3 - 0 = 3$$

$$A_4 = \int_1^2 (12 - 3x^2)\,dx = 12x - x^3\Big]_1^2$$

$$= (24 - 8) - (12 - 1) \quad = 5$$

$$\therefore\ 2(A_3 + A_4) = 2(3 + 5) = 16$$

$$\therefore\ \text{Enclosed Area } = 32 - 16 = 16$$



Blunders (-3)
B1 Indices
B2 Integration
B3 Calculating roots of *f(x)=0*
B4 Calculating $f \cap g$
B5 Error in area formula
B6 Incorrect order in applying limits
B7 Not calculating substituted limits
B8 Error with *f(x)* or *g(x)*
B9 Uses $\pi\int y\,dx$ for area formula

Attempts
A1 Uses volume formula
A2 Uses y^2 in formula

Worthless
W1 Differentiation instead of integration except where other work merits attempt
W2 Wrong area formula and no work

MARKING SCHEME

LEAVING CERTIFICATE EXAMINATION 2006

MATHEMATICS – HIGHER LEVEL – PAPER 2

GENERAL GUIDELINES FOR EXAMINERS – PAPER 2

1. Penalties of three types are applied to candidates' work as follows:
   - Blunders - mathematical errors/omissions (-3)
   - Slips - numerical errors (-1)
   - Misreadings (provided task is not oversimplified) (-1).

   Frequently occurring errors to which these penalties must be applied are listed in the scheme. They are labelled: B1, B2, B3,…, S1, S2,…, M1, M2,…etc. These lists are not exhaustive.

2. When awarding attempt marks, e.g. Att(3), note that
   - any *correct, relevant* step in a part of a question merits at least the attempt mark for that part
   - if deductions result in a mark which is lower than the attempt mark, then the attempt mark must be awarded
   - a mark between zero and the attempt mark is never awarded.

3. Worthless work is awarded zero marks. Some examples of such work are listed in the scheme and they are labelled as W1, W2,…etc.

4. The phrase "hit or miss" means that partial marks are not awarded – the candidate receives all of the relevant marks or none.

5. The phrase "and stops" means that no more work is shown by the candidate.

6. Special notes relating to the marking of a particular part of a question are indicated by an asterisk. These notes immediately follow the box containing the relevant solution.

7. The sample solutions for each question are not intended to be exhaustive lists – there may be other correct solutions. Any examiner unsure of the validity of the approach adopted by a particular candidate to a particular question should contact his/her advising examiner.

8. Unless otherwise indicated in the scheme, accept the best of two or more attempts – even when attempts have been cancelled.

9. The *same* error in the *same* section of a question is penalised *once* only.

10. Particular cases, verifications and answers derived from diagrams (unless requested) qualify for attempt marks at most.

11. A serious blunder, omission or misreading results in the attempt mark at most.

12. Do not penalise the use of a comma for a decimal point, e.g. €5.50 may be written as €5,50.

QUESTION 1

Part (a)	**15 marks**	**Att 5**
Part (b)	**20 (10, 10) marks**	**Att (3, 3)**
Part (c)	**15 (10,5) marks**	**Att (3, 2)**

.

Part (a) **15 marks** **Att 5**

1 (a) a (–1, –3) and b (3, 1) are the end-points of a diameter of a circle.
Write down the equation of the circle.

1 (a) Mid-point of $[ab]$ = Centre of circle $c = (1, -1)$

$\text{Radius} = |ac| = \sqrt{4+4} = \sqrt{8}.$

$\therefore \text{Equation of circle}: (x-1)^2 + (y+1)^2 = 8.$

Blunders
B1 Error in mid-point formula (apply once).
B2 Error in distance formula (apply once)
B3 Incorrect substitution for each formula.

Slips
S1 Arithmetic errors

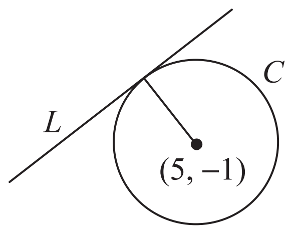
Attempts
A1 Radius or midpoint only
A2 General form of equation written down and some correct substitution

Part (b) **20 (10, 10) marks** **Att (3, 3)**

(b) Circle C has centre (5, –1).
The line L: $3x - 4y + 11 = 0$ is a tangent to C.

(i) Show that the radius of C is 6.

(ii) The line $x + py + 1 = 0$ is also a tangent to C.
Find two possible values of p.



Part (b) (i) **10 marks** **Att 3**

1 (b) (i)
Radius = Distance from centre (5, –1) to line $3x - 4y + 11 = 0$.

$$\text{Radius} = \left|\frac{15+4+11}{\sqrt{9+16}}\right| = 6.$$

Blunders
B1 Error in distance formula

Slips
S1 Arithmetic errors

Attempts
A1 Picks point on 3x - 4y +11 = 0 and finds distance using $\sqrt{(x_2 - x_1)^2 - (y_2 - y_1)^2}$

Part (b) (ii) **10 marks** **Att 3**

1 (b) (ii) Perpendicular distance from centre (5, –1) to $x + py + 1 = 0$ equals 6.

$$\therefore \left| \frac{5 - p + 1}{\sqrt{1 + p^2}} \right| = 6 \quad \Rightarrow \quad (6 - p)^2 = 36 + 36p^2.$$

$$\therefore\ 35p^2 + 12p = 0 \quad \Rightarrow \quad p(35p + 12) = 0 \quad \Rightarrow \quad p = 0 \text{ or } p = -\frac{12}{35}$$

Blunders

B1 Error in perpendicular distance formula
B2 Error in squaring or fails to square
B3 Error in solving quadratic

Slips

S1 Arithmetic

Attempts

A1 Attempts to substitute and solve for p

Part (c) **15 (10,5) marks** **Att (3,2)**

1 (c) *S* is the circle $x^2 + y^2 + 4x + 4y - 17 = 0$ and *K* is the line $4x + 3y = 12$.
(i) Show that the line *K* does not intersect *S*.
(ii) Find the co-ordinates of the point on *S* that is closest to *K*.

Part (c) (i) **10 marks** **Att 3**

1 (c) (i)

Centre = (–2, –2), radius $= \sqrt{g^2 + f^2 - c} = \sqrt{4 + 4 + 17} = 5$.

Distance from line to centre $= \left| \frac{-8 - 6 - 12}{5} \right| = \frac{26}{5} > 5$. $\therefore$ *K* does not intersect *S*.

Blunders

B1 Error centre or radius formula (each formula)
B2 Error in perpendicular distance formula (mod omitted)
B3 Error in squaring
B4 No conclusion

Slips

S1 Arithmetic

Part (c) (ii) **5 marks** **Att 2**

1 (c) (ii)	Required point is on line containing centre and perpendicular to $4x+3y=12$.
	$y+2=\frac{3}{4}(x+2) \Rightarrow 3x-4y=2$. Required point is $3x-4y=2 \cap S$.
	$x=\frac{2+4y}{3} \Rightarrow \left(\frac{2+4y}{3}\right)^2+y^2+4\left(\frac{2+4y}{3}\right)+4y-17=0$.
	$\therefore \frac{4+16y+16y^2}{9}+y^2+\frac{8+16y}{3}+4y-17=0$.
	$4+16y+16y^2+9y^2+24+48y+36y-153=0 \Rightarrow 25y^2+100y-125=0$.
	$\therefore y^2+4y-5=0 \Rightarrow (y-1)(y+5)=0 \Rightarrow y=1$ or $y=-5$.
	$\therefore (2,1)$ or $(-6,-5)$. But $(2,1)$ is closest point. $\therefore$ Solution $=(2,1)$.

Blunders
B1 Error in equation of line
B2 Error in substitution
B3 Error in squaring
B4 Error in forming quadratic
B5 Error in solving quadratic
B5 No conclusion

Slips
S1 Arithmetic

Attempts
A1 Writes down the centre or the radius

QUESTION 2

Part (a)	**10 marks**	**Att 3**
Part (b)	**30 ([5,5],10,10) marks**	**Att (2,2,3,3)**
Part (c)	**10(5,5)marks**	**Att (2,2)**

Part (a) **10 marks** **Att 3**

2 (a) $\vec{x} = -3\vec{i} + \vec{j}$. Express $\left(\vec{x}^{\perp}\right)^{\perp}$ in terms of $\vec{i}$ and $\vec{j}$.

Part (a) **10 marks** **Att 3**

2(a)

$$\vec{x} = -3\vec{i} + \vec{j} \Rightarrow \vec{x}^{\perp} = \left(-3\vec{i} + \vec{j}\right)^{\perp} = -\vec{i} - 3\vec{j} \Rightarrow \left(\vec{x}^{\perp}\right)^{\perp} = 3\vec{i} - \vec{j}.$$

Blunders
B1 Incorrect sign
B2 Incorrect scalar

Attempts
A1 Correct formula

Part (b) **30 (5, 5, 10, 10) marks** **Att (2, 2, 3, 3)**

(b) $\vec{p} = -5\vec{i} + 2\vec{j}$, $\vec{q} = \vec{i} - 6\vec{j}$ and $\vec{r} = -\vec{i} + 5\vec{j}$.

(i) Express $\overrightarrow{pq}$ and $\overrightarrow{pr}$ in terms of $\vec{i}$ and $\vec{j}$.

(ii) Given that $10\vec{s} = \left|\overrightarrow{pr}\right|\overrightarrow{pq} + \left|\overrightarrow{pq}\right|\overrightarrow{pr}$, express $\vec{s}$ in terms of $\vec{i}$ and $\vec{j}$.

(iii) Find the measure of the angle between $\vec{s}$ and $\overrightarrow{pr}$.

(i) Express $\overrightarrow{pq}$ **5 marks** **Att 2**

Express $\overrightarrow{pr}$ **5 marks** **Att 2**

2 (b) (i)

$$\overrightarrow{pq} = \vec{q} - \vec{p} = \vec{i} - 6\vec{j} + 5\vec{i} - 2\vec{j} = 6\vec{i} - 8\vec{j}.$$

$$\overrightarrow{pr} = \vec{r} - \vec{p} = -\vec{i} + 5\vec{j} + 5\vec{i} - 2\vec{j} = 4\vec{i} + 3\vec{j}.$$

Blunders
B1 $\overrightarrow{pq} \neq \vec{q} - \vec{p}$ or $\overrightarrow{pr} \neq \vec{r} - \vec{p}$
B2 Error in signs

Slips
S1 Arithmetic

Part (b) (ii) **10 marks** **Att 3**

2 (b) (ii)

$$\left|\overrightarrow{pr}\right| = \left|4\vec{i}+3\vec{j}\right| = \sqrt{16+9} = 5 \text{ and } \left|\overrightarrow{pq}\right| = \left|6\vec{i}-8\vec{j}\right| = \sqrt{36+64} = 10.$$

$$\therefore\ 10\vec{s} = 5\left(6\vec{i}-8\vec{j}\right)+10\left(4\vec{i}+3\vec{j}\right) \Rightarrow 2\vec{s} = 6\vec{i}-8\vec{j}+8\vec{i}+6\vec{j} = 14\vec{i}-2\vec{j}.$$

$$\therefore\ \vec{s} = 7\vec{i}-\vec{j}.$$

Blunders

B1 Error in $\left|\overrightarrow{pq}\right|$ or in $\left|\overrightarrow{pr}\right|$

Slips

S1 Arithmetic

Part (b) (iii) **10 marks** **Att 3**

2 (b) (iii)

$$\vec{s}\,.\,\overrightarrow{pr} \Rightarrow \left(7\vec{i}-\vec{j}\right)\left(4\vec{i}+3\vec{j}\right) = \left|7\vec{i}-\vec{j}\right|.\left|4\vec{i}+3\vec{j}\right|\cos\theta.$$

$$\therefore\ \cos\theta = \frac{28-3}{\sqrt{50}\sqrt{25}} = \frac{25}{25\sqrt{2}} = \frac{1}{\sqrt{2}}. \quad \therefore\ \theta = \frac{\pi}{4}.$$

Blunders

B1 Error in form of scalar product each time

B2 Error in calculating $\cos^{-1}\frac{1}{\sqrt{2}}$

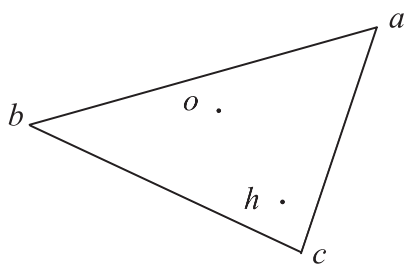
Slips

S1 Arithmetic

Part (c) **10 (5, 5) marks** **Att (2, 2)**

(c) The origin *o* is the circumcentre of the triangle *abc*.

If $\vec{h} = \vec{a} + \vec{b} + \vec{c}$, show that $\overrightarrow{ah} \perp \overrightarrow{bc}$.



Simplify $\overrightarrow{ah}$ and $\overrightarrow{bc}$ **5 marks** **Att 2**

Finish **5 marks** **Att 2**

2 (c)

$$\overrightarrow{ah}.\overrightarrow{bc} = \left(\vec{h} - \vec{a}\right)\left(\vec{c} - \vec{b}\right) = \left(\vec{b} + \vec{c}\right)\left(\vec{c} - \vec{b}\right) = \left|\vec{c}\right|^2 - \left|\vec{b}\right|^2 .$$

But since *o* is the circumcentre, $\left|\vec{b}\right| = \left|\vec{c}\right|$. $\therefore\ \overrightarrow{ah}.\overrightarrow{bc} = 0 \Rightarrow \overrightarrow{ah} \perp \overrightarrow{bc}$.

Blunders

B1 $\overrightarrow{ah} \neq \vec{h} - \vec{a}$

B2 $\overrightarrow{bc} \neq \vec{c} - \vec{b}$

B3 Error in vector multiplication

Slips

S1 Arithmetic errors

Attempts

A1 States condition for perpendicular vectors correctly

A2 $\overrightarrow{ah} = \vec{h} - \vec{a}$

QUESTION 3

Part (a)	**15 marks**	**Att 5**
Part (b)	**10 marks**	**Att 3**
Part (c)	**25 (10, 15) marks**	**Att 3,5**

Part (a) **15 marks** **Att 5**

3 (a) Show that the line containing the points (3, –6) and (–7, 12) is perpendicular to the line $5x - 9y + 6 = 0$.

Part (a) **15 marks** **Att 5**

3 (a) Slope of line containing points (3, –6) and (–7, 12) is $m_1 = \frac{12+6}{-7-3} = \frac{18}{-10} = -\frac{9}{5}$.

The line $5x - 9y + 6 = 0$ has slope $m_2 = \frac{5}{9}$.

But $m_1.m_2 = -1$, $\therefore$ lines perpendicular.

Blunders
B1 Error in finding either slope
B2 Product of slopes not shown = -1
B3 No conclusion

Slips
S1 Arithmetic

Part (b) **10 marks** **Att 3**

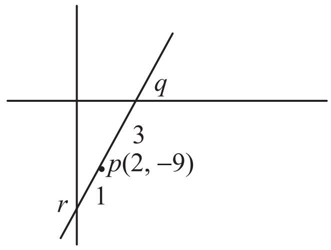
3 (b) The line *K* has positive slope and passes through the point *p* (2, –9).
K intersects the *x*-axis at *q* and the *y*-axis at *r* and $|pq| : |pr| = 3:1$.
Find the co-ordinates of *q* and the co-ordinates of *r*.

3 (b)

Equation of $K: y + 9 = m(x-2)$. q is $\left(\frac{9}{m} + 2, 0\right)$ and r is $(0, -2m-9)$.

$$\therefore \frac{\frac{9}{m} + 2 + 0}{4} = 2 \Rightarrow \frac{9}{m} + 2 = 8 \Rightarrow \frac{9}{m} = 6 \Rightarrow m = \frac{3}{2}.$$

$\therefore$ q is $(8, 0)$ and r is $(0, -12)$.



Blunders
B1 Error in finding equation of line
B2 Error in finding q or r each time
B3 Error in ratio formula
B4 Error in simplification if not a slip

Slips
S1 Arithmetic errors

Attempts
Correct formula for K for ratio for distance with some correct substitution

Part (c) **25(10, 15) marks** **Att (3, 5)**

3 (c) (i) Prove that the measure of one of the angles between two lines with slopes m_1 and m_2 is given by $\tan\theta = \dfrac{m_1 - m_2}{1 + m_1 m_2}$.

(ii) L is the line $y = 4x$ and K is the line $x = 4y$.
f is the transformation $(x, y) \to (x', y')$, where $x' = 2x - y$ and $y' = x + 3y$.
Find the measure of the acute angle between $f(L)$ and $f(K)$, correct to the nearest degree.

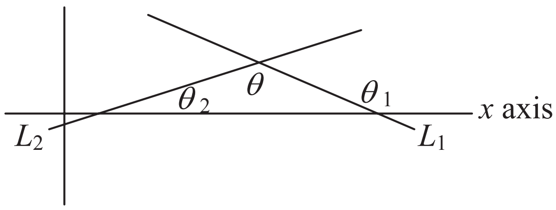
Part (c) (i) **10 marks** **Att 3**

3 (c) (i)

Slope $L_1 = m_1 = \tan\ \theta_1$.
Slope $L_2 = m_2 = \tan\ \theta_2$.
$\theta_1 = \theta + \theta_2 \ \Rightarrow\ \theta = \theta_1 - \theta_2$.

$$\therefore \tan\theta = \tan(\theta_1 - \theta_2) = \frac{\tan\theta_1 - \tan\theta_2}{1 + \tan\theta_1 \tan\theta_2}$$

$$\therefore\ \tan\theta = \frac{m_1 - m_2}{1 + m_1 m_2}.$$



Blunders
B1 Error in expressing q in terms of q_1 *and* q_2
B2 Error in tan $(\theta_1 - \theta_2)$

Part (c) (ii) **15 marks** **Att 5**

3 (c) (ii)

$3x' = 6x - 3y$
$y' = x + 3y$

$7x = 3x' + y' \ \Rightarrow\ x = \frac{1}{7}(3x' + y')$. But $y' = x + 3y \ \Rightarrow\ y' = \frac{1}{7}(3x' + y') + 3y$.

$\therefore\ y = \frac{1}{7}(-x' + 2y')$.

$f(L): \frac{1}{7}(-x' + 2y') = \frac{4}{7}(3x' + y')$. $\therefore\ f(L): 2y' = -13x' \ \Rightarrow$ slope $f(L) = -\frac{13}{2}$.

$f(K): \frac{1}{7}(3x' + y') = \frac{4}{7}(-x' + 2y')$. $\therefore\ f(K): y' = x' \ \Rightarrow$ slope $f(K) = 1$.

$$\tan\ \theta = \frac{-\frac{13}{2} - 1}{1 - \frac{13}{2}} = \frac{15}{11} \ \Rightarrow\ \theta = 54^{\circ}.$$

Blunders
B1 Error in setting up/solving simultaneous equations each time
B2 Error in calculating slope for F(L) and F(K) each time for different blunder
B3 Error in applying formula
B4 Error in calculating $\tan^{-1}\frac{15}{11}$

Slips
S1 Arithmetic errors

QUESTION 4

Part (a)	**15 marks**	**Att 5**
Part (b)	**20 (10, 10) marks**	**Att (3,3)**
Part (c)	**15 (10,5) marks**	**Att (3,2)**

Part (a) **15 marks** **Att 5**

4 (a) Write down the values of A for which $\cos A = \frac{1}{2}$, where $0^{\circ} \leq A \leq 360^{\circ}$.

4 (a) $\cos A = \frac{1}{2} \Rightarrow A = 60^{\circ}, 300^{\circ}$.

Blunders

B1 Each incorrect or omitted value

Part (b) **20 (10, 10) marks** **Att (3,3)**

4 (b) (i) Express $\sin(3x + 60^{\circ}) - \sin x$ as a product of sine and cosine.

(ii) Find all the solutions of the equation

$$\sin(3x + 60^{\circ}) - \sin x = 0, \text{ where } 0^{\circ} \leq x \leq 360^{\circ}.$$

Part (b) (i) **10 marks** **Att 3**

4 (b) (i) $\sin(3x + 60^{\circ}) - \sin x = 2\cos(2x + 30^{\circ})\sin(x + 30^{\circ})$.

Blunders

B1 Error in simplifying the expression

Part (b) (ii) **10 marks** **Att 3**

4 (b) (ii)

$$\sin(3x + 60^{\circ}) - \sin x = 0 \quad \Rightarrow \quad 2\cos(2x + 30^{\circ})\sin(x + 30^{\circ}) = 0$$

$$\therefore \cos(2x + 30^{\circ}) = 0 \quad \text{or} \quad \sin(x + 30^{\circ}) = 0$$

$$2x + 30^{\circ} = 90^{\circ}, 270^{\circ}, 450^{o}, 630^{\circ} \quad \text{or} \quad x + 30^{\circ} = 180^{\circ}, 360^{\circ}.$$

$$\therefore x = 30^{\circ}, 120^{\circ}, 210^{\circ}, 300^{0} \quad \text{or} \quad x = 150^{\circ}, 330^{\circ}.$$

$$\therefore \text{ Solution} = \{30^{\circ}, 120^{\circ}, 150^{\circ}, 210^{\circ}, 300^{\circ}\ 330^{\circ}\}.$$

Blunders

B1 Error in solving

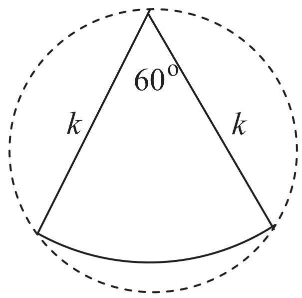
B2 Each incorrect or missing solution

Part (c) **15 (10, 5) marks** **Att (3, 2)**

4 (c) The diagram shows a sector (solid line) circumscribed by a circle (dashed line).

(i) Find the radius of the circle in terms of *k*.

(ii) Show that the circle encloses an area which is double that of the sector.

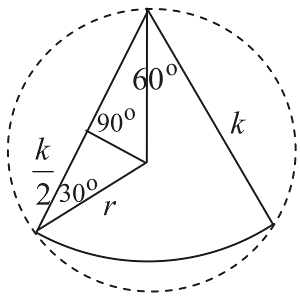


Part (c) (i) **10 marks** **Att 3**

4 (c) (i)

$$\cos 30^0 = \frac{\frac{k}{2}}{r} \quad \Rightarrow \quad \frac{r\sqrt{3}}{2} = \frac{k}{2}$$

$$\Rightarrow \quad r = \frac{k}{\sqrt{3}}.$$



Blunders

B1 Incorrect use of cosine rule or sine rule or area of triangle.

B2 Error in cos from right-angled triangle

Part (c) (ii) **5 marks** **Att 2**

4 (c) (ii)

$$\text{Area of circle } = \pi r^2 = \frac{\pi k^2}{3}.$$

$$\text{Area of sector } = \frac{1}{2}k^2\theta = \frac{k^2}{2}.\frac{\pi}{3} = \frac{\pi k^2}{6}.$$

$\therefore$ Area of circle = 2× area of sector.

Blunders

B1 Error in area of sector

B2 Error in area of circle

B3 No conclusion

Slips

S1 Arithmetic

QUESTION 5

Part (a)	**25 (10, 5, 5, 5) marks**	**Att (3, 2, 2, 2)**
Part (b)	**25 (15, 10) marks**	**Att (5, 3)**

Part (a) **25 (10, 5, 5, 5) marks** **Att (3, 2, 2, 2)**

5 (a) (i) Copy and complete the table below for $f : x \to \tan^{-1} x$, giving the values for $f(x)$ in terms of π.

x	$-\sqrt{3}$	-1	$-\frac{1}{\sqrt{3}}$	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$
$f(x)$						$\frac{\pi}{4}$	

(ii) Draw the graph of $y = f(x)$ in the domain $-2 \le x \le 2$, scaling the y-axis in terms of π.

(iii) Draw the two horizontal asymptotes of the graph.

(iv) For some values of $k \in \mathbf{R}$, but not all values, $\tan^{-1}(\tan k) = k$.

State the range of values of k for which $\tan^{-1}(\tan k) = k$.

Show, by means of an example, what happens outside the range.

Part (a) (i) **10 marks** **Att 3**

5 (a) (i)

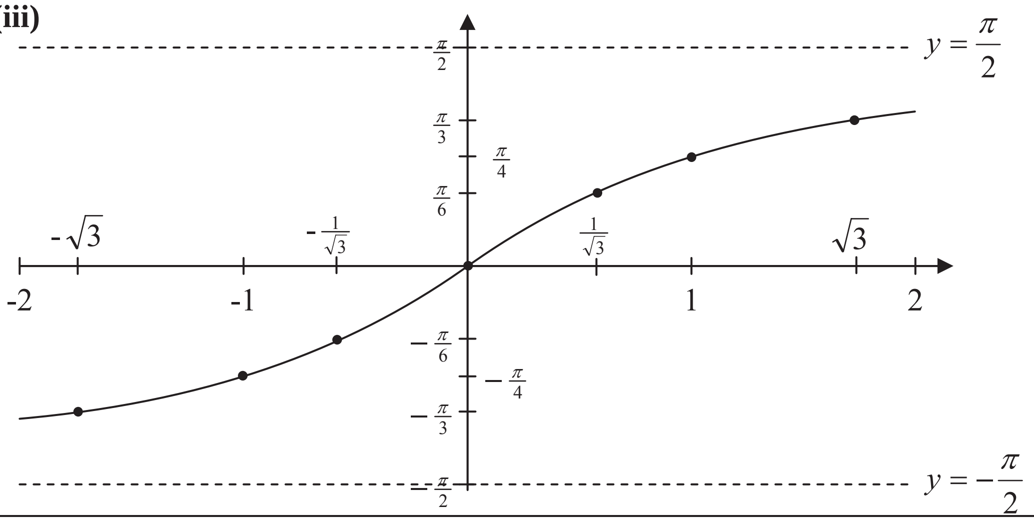
x	$-\sqrt{3}$	-1	$-\frac{1}{\sqrt{3}}$	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$
$f(x)$	$-\frac{\pi}{3}$	$-\frac{\pi}{4}$	$-\frac{\pi}{6}$	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$

Mark as follows:

No. of values correct	1	2	3	4	5	6
Mark	Att	Att	7	8	9	10

Part (a) (ii) **5 marks** **Att 2**
Part (a) (iii) **5 marks** **Att 2**

5 (a) (ii) & (iii)



Blunders
B1 Asymptotes not horizontal
B2 Asymptotes do not contain $-\dfrac{\pi}{2}$ and $\dfrac{\pi}{2}$
Slips
S1 Each incorrectly plotted point

Part (a) (iv) **5 marks** **Att 2**

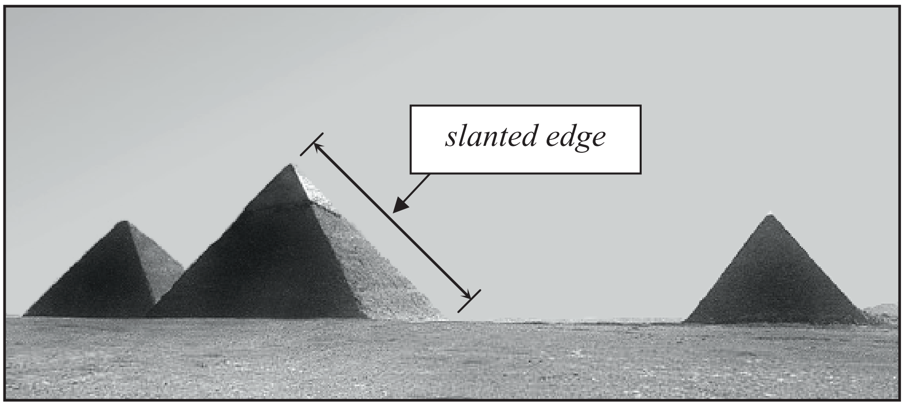
5 (a) (iv) Range of values: $-\dfrac{\pi}{2} < k < \dfrac{\pi}{2}$.

e.g. if $k = \dfrac{5\pi}{4}$, then $\tan^{-1}\left(\tan\dfrac{5\pi}{4}\right) = \tan^{-1}1 = \dfrac{\pi}{4} \neq \dfrac{5\pi}{4}$.

Blunders
B1 Incorrect endpoint of range each time
B2 No example given

Part (b) **25 (15, 10) marks** **Att (5, 3)**

5 (b) The great pyramid at Giza in Egypt has a square base and four triangular faces.
The base of the pyramid is of side 230 metres and the pyramid is 146 metres high.
The top of the pyramid is directly above the centre of the base.



(i) Calculate the length of one of the slanted edges, correct to the nearest metre.
(ii) Calculate, correct to two significant figures, the total area of the four triangular faces of the pyramid (assuming they are smooth flat surfaces).

Part (b) (i) **15 marks** **Att 5**

5 (b) (i) Diagonal of square base $= d = \sqrt{230^2 + 230^2} = \sqrt{105800}$.

Let length of slant edge = s and height of pyramid = h.

$$s^2 = h^2 + \left(\frac{1}{2}d\right)^2 \quad \Rightarrow \quad s = \sqrt{21316 + 26450} = \sqrt{47766} = 218.5 = 219 \text{ metres.}$$

Blunders

B1 Incorrect application of Pythagoras each time or incorrect trig ratio

Slips

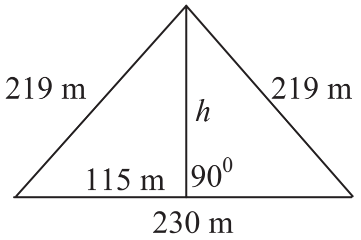
S1 Error in calculations

Part (b) (ii) **10 marks** **Att 3**

5 (b) (ii)

$$h = \sqrt{219^2 - 115^2} = \sqrt{47961 - 13225}$$

$$h = \sqrt{34736} = 186.37.$$



Total surface area = $4 \times \frac{1}{2}(230).186.37 = 85730.2 = 86000 \text{ m}^2$.

Blunders

B1 Error in Pythagoras or in area of triangle formula

Slips

Arithmetic errors or failure to round off.

QUESTION 6

Part (a)	**10 (5, 5) marks**	**Att (- , 2)**
Part (b)	**25 (5, 5, 5, 5,5) marks**	**Att (2, 2, 2, 2,2)**
Part (c)	**15 (5, 5, 5) marks**	**Att (2, 2, 2)**

Part (a) **10 (5, 5) marks** **Att (- , 2)**

6 (a) **(i)** How many different teams of three people can be chosen from a panel of six boys and five girls?

(ii) If the team is chosen at random, find the probability that it consists of girls only?

Part (a) (i) **5 marks** **Hit/Miss**

6 (a) (i) Answer $= {}^{11}C_3 = 165.$

Part (a) (ii) **5 marks** **Att 2**

6 (a) (ii) 3 girls to choose from 5 $\Rightarrow$ Solution $= {}^{5}C_3 = 10.$

$$P(\text{all girls}) = \frac{10}{165} = \frac{2}{33}$$

Blunders

B1 Incorrect total possible

B2 Incorrect total favourable

Slips

S1 Arithmetic errors

Part (b) **25 (5, 5, 5, 5, 5) marks** **Att (2, 2, 2, 2, 2)**

6 (b) **(i)** Solve the difference equation $6u_{n+2} - 7u_{n+1} + u_n = 0,$ where $n \geq 0,$ given that $u_o = 8$ and $u_1 = 3.$

(ii) Verify that the solution to part **(i)** also satisfies the difference equation

$$6u_{n+1} - u_n - 10 = 0.$$

(b) (i) Char. Eqn.	**5 marks**	**Att 2**
Roots	**5 marks**	**Att 2**
Sim. Eqns.	**5 marks**	**Att 2**
Finish	**5 marks**	**Att 2**

6 (b) (i)

$$6u_{n+2} - 7u_{n+1} + u_n = 0.$$

$$\therefore\ 6x^2 - 7x + 1 = 0 \quad \Rightarrow \quad (x-1)(6x-1) = 0 \quad \Rightarrow \quad x = 1 \text{ or } x = \frac{1}{6}.$$

$$\therefore\ u_n = l(1)^n + m\left(\frac{1}{6}\right)^n = l + m\left(\frac{1}{6}\right)^n.$$

$$u_o = 8 \ \Rightarrow\ l + m = 8 \ \text{ and } \ u_1 = 3 \ \Rightarrow\ l + \frac{1}{6}m = 3.$$

$$\frac{5}{6}m = 5 \ \Rightarrow\ m = 6 \text{ and } l = 2. \ \therefore\ u_n = 2 + 6\left(\frac{1}{6}\right)^n = 2 + \left(\frac{1}{6}\right)^{n-1}.$$

Blunders
B1 Error in characteristic equation
B2 Error in factors or quadratic formula
B3 Incorrect use of initial conditions

Slips
S1 Arithmetic errors

Attempts
A1 Char equation
A2 Eqn in *l* and *m*

Part (b) (ii) **5 marks** **Att 2**

6 (b) (ii)

$$u_n = 2 + \left(\frac{1}{6}\right)^{n-1} \text{ and } 6u_{n+1} - u_n - 10 = 0.$$

$$\therefore\ 6\left[2 + \left(\frac{1}{6}\right)^{n}\right] - \left[2 + \left(\frac{1}{6}\right)^{n-1}\right] - 10 = 12 + \left(\frac{1}{6}\right)^{n-1} - 2 - \left(\frac{1}{6}\right)^{n-1} - 10 = 0. \quad \therefore \text{ solution.}$$

Blunders
B1 Error in U_{n+1} or U_n
B2 Error in indices

Part (c) **15 (5, 5, 5) marks** **Att (2, 2, 2)**

6 (c) There are thirty days in June. Seven students have their birthdays in June.
The birthdays are independent of each other and all dates are equally likely.
(i) What is the probability that all seven students have the same birthday?
(ii) What is the probability that all seven students have different birthdays?
(iii) Show that the probability that at least two have the same birthday is greater that 0·5.

Part (c) (i) **5 marks** **Att 2**

6 (c) (i)

$$P=\frac{\text{total favourable}}{\text{total possible}}=\frac{30\times1\times1\times1\times1\times1\times1}{30\times30\times30\times\times30\times30\times30\times30}=\frac{1}{(30)^6}$$

Blunders
B1 Incorrect total possible
B2 Incorrect total favourable
B3 No fraction

Part (c) (ii) **5 marks** **Att 2**

6 (c) (ii)

$$P=\frac{\text{total favourable}}{\text{total possible}}=\frac{30\times29\times28\times27\times26\times25\times24}{30\times30\times30\times\times30\times30\times30\times30}=\frac{2639}{5625}$$

Blunders
B1 Incorrect total possible
B2 Incorrect total favourable
B3 No fraction

Part (c) (iii) **5 marks** **Att 2**

6 (c) (iii)

$$\text{Probability} = 1-\text{P(all seven have different birthdays)}$$

$$=1-\frac{29\times28\times27\times26\times25\times24}{(30)^6}=1-0.4691=0.5309>0.5.$$

Blunders
B1 Error in correct total possible
B2 Error in correct total favourable
B3 No conclusion

QUESTION 7

Part (a)	**10 (5, 5) marks**	**Att (2, 2)**
Part (b)	**25 (5,10, 5, 5) marks**	**Att (- , 3, 2, 2)**
Part (c)	**15 marks**	**Att 5**

Part (a)	**10 (5, 5) marks**	**Att (2, 2)**

7 (a) The password for a mobile phone consists of five digits.
- **(i)** How many passwords are possible?
- **(ii)** How many of these passwords start with a 2 and finish with an odd digit?

Part (a) (i)	**5 marks**	**Att 2**

7 (a) (i) Number of possible passwords $= 10^5 = 100,000$

Slips

S1 Gives $9^5 = 59049$

Part (a) (ii)	**5 marks**	**Att 2**

7 (a) (ii) Number of passwords $= 1 \times 10^3 \times 5 = 5,000$.

Blunders

B1 Adds $1 + 10^3 + 5$

Part (b)	**25 (5, 10, 5, 5) marks**	**Att (- , 3, 2, 2)**

7 (b) For a lottery, 35 cards numbered 1 to 35 are placed in a drum.
Five cards will be chosen at random from the drum as a winning combination.
- **(i)** How many different combinations are possible?
- **(ii)** How many of all the possible combinations will match exactly four numbers with the winning combination?
- **(iii)** How many of all the possible combinations will match exactly three numbers with the winning combination?
- **(iv)** Show that the probability of matching at least three numbers with the winning combination is approximately 0·014.

Part (b) (i)	**5 marks**	**Hit/Miss**

7 (b) (i) Number of different possible combinations $= {}^{35}C_5 = 324,632$.

Part (b) (ii)	**10 marks**	**Att 3**

7 (b) (ii) Match four $= {}^{5}C_4 \times {}^{30}C_1 = 150$.

Blunders

B1 Addition for multiplication

B2 31 for 30

Part (b) (iii)	5 marks	Att 2

7 (b) (iii) Match three $= {}^5C_3 \times {}^{30}C_2 = 4350.$

Blunders

B1 Addition for multiplication
B2 32 for 30

Part (b) (iv)	5 marks	Att 2

7 (b) (iv) P(of matching at least three numbers)
= P(matching three) + P(matching four)+ P(matching five)

$$= \frac{4350+150+1}{324632} = \frac{4501}{324632} = 0.01386493 = 0.014.$$

Blunders

B1 Error in total favourable
B2 Error in total possible
B3 No fraction
B4 Incorrect or no conclusion

Part (c)	15 marks	Att 5

7 (c) The mean of the integers from $-n$ to n, inclusive, is 0.

Show that the standard deviation is $\sqrt{\frac{n(n+1)}{3}}$.

Part (c)	15 marks	Att 5

7 (c)

$$\sigma^2 = \frac{(-n)^2+(-n+1)^2+......(-2)^2+(-1)^2+(0)^2+(1)^2+(2)^2.....(n-2)^2+(n-1)^2+(n)^2}{2n+1}.$$

$$\therefore\ \sigma^2 = \frac{2\left[1^2+2^2+3^2+......+n^2\right]}{2n+1} = \frac{2}{2n+1}\times\frac{n}{6}(n+1)(2n+1) = \frac{n(n+1)}{3}.$$

$$\therefore\ \sigma = \sqrt{\frac{n(n+1)}{3}}.$$

Blunders

B1 Not squared
B2 No square root
B3 Mean not found or incorrect denominator
B5 Error in sum to n terms

QUESTION 8

Part (a)	15 marks	Att 5
Part (b)	15 (10, 5) marks	Att (3,2)
Part (c)	20 (10, 10) marks	Att (3, 3)

Part (a) **15 marks** **Att 5**

8 (a) Derive the Maclaurin series for $f(x) = e^x$ up to and including the term containing x^3.

8 (a)

$$f(x) = f(0) + \frac{f'(0)x}{1!} + \frac{f''(0)x^2}{2!} + \frac{f'''(0)x^3}{3!} + \ldots\ldots\ldots\ldots.$$

$$f(x) = e^x \quad \Rightarrow \quad f(0) = e^0 = 1.$$

$$f'(x) = e^x \quad \Rightarrow \quad f'(0) = 1.$$

$$f''(x) = e^x \quad \Rightarrow \quad f''(0) = 1.$$

$$f'''(x) = e^x \quad \Rightarrow \quad f'''(0) = 1.$$

$$\therefore f(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots\ldots\ldots\ldots.$$

Blunders

B1 Error in differentiation each time if error not consistent

B2 Error in evaluating $f^{(n)}(0)$

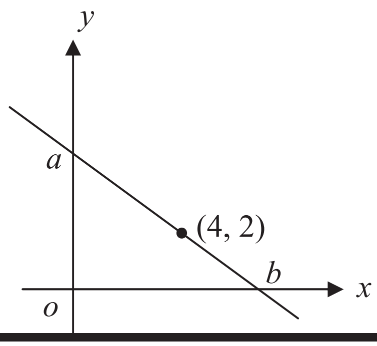
B3 Each term missing

Part (b) **15 (10, 5) marks** **Att (3,2)**

8 (b) A line passes through the point (4, 2) and has slope m, where $m < 0$. The line intersects the axes at the points a and b.

(i) Find the co-ordinates of a and b, in terms of m.

(ii) Hence, find the value of m for which the area of triangle aob is a minimum.



Part (b) (i) **10 marks** **Att 3**

8 (b) (i) Equation of line $y - 2 = m(x - 4) \Rightarrow mx - y = 4m - 2.$

$$\therefore \quad a(0, -4m + 2) \text{ and } b\left(4 - \frac{2}{m}, 0\right).$$

Blunders

B1 Error in finding the equation of the line

B2 a and b coordinates must have 0 in correct position

B3 Not in coordinate form

Part (b) (ii) **5 marks** **Att2**

8 (b) (ii)

$$a\left(0, -4m+2\right),\ b\left(4-\frac{2}{m}, 0\right).$$

Area of triangle = $A = \frac{1}{2}\left(4-\frac{2}{m}\right)\left(-4m+2\right).$

$$A = \frac{1}{2}\left(-16m+8+8-\frac{4}{m}\right) = \frac{1}{2}\left(-16m+16-4m^{-1}\right) = -8m+8-2m^{-1}.$$

$\frac{dA}{dm} = -8+2m^{-2} = 0,$ for minimum.

$\therefore\ -4+\frac{1}{m^2} = 0 \quad \Rightarrow \quad 4m^2 = 1 \quad \Rightarrow \quad m = -\frac{1}{2}$ as $m < 0.$

.

$\therefore\ A = -8\left(-0.5\right)+8-\frac{2}{-0.5} = 4+8+4 = 16,$ minimum area.

Blunders

B1 Error in formula for area of a triangle
B2 Error in derivative
B3 Error in solving equation

Slips

S1 Arithmetic errors

Part (c) **20 (10, 10) marks** **Att (3, 3)**

8 (c) Use the ratio test to test each of the following series for convergence.
In each case, specify clearly the range of values of x for which the series converges, the range of values for which it diverges, and the value(s) of x for which the test is inconclusive.

(i) $\sum_{n=1}^{\infty} n3^n x^n$ **(ii)** $\sum_{n=1}^{\infty} \frac{(n+1)!n!}{(2n)!}x^n.$

Part (c) (i) **10 marks** **Att 3**

$$u_n = n3^n x^n \quad \Rightarrow \quad u_{n+1} = (n+1)3^{n+1}x^{n+1}.$$

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)3^{n+1}x^{n+1}}{n3^n x^n}\right| = \lim_{n\to\infty}\left|3x\left(1+\frac{1}{n}\right)\right| = |3x|.$$

Converges for $|3x| < 1 \Rightarrow -\frac{1}{3} < x < \frac{1}{3}.$

Diverges for $|3x| > 1 \Rightarrow x > \frac{1}{3}$ or $x < -\frac{1}{3}.$

Inconclusive for $|3x| = 1$ i.e. $x = \pm\frac{1}{3}.$

Blunders

B1 Error in U_{n+1}

B2 Error in evaluating $\frac{U_{n+1}}{U_n}$

B3 Error in evaluating limit

B4 Range incorrectly or not applied

Misreading (-1) for each case omitted

Part (c) (ii) **10 marks** **Att 3**

8 (c) (ii) $\sum_{n=1}^{\infty} \frac{(n+1)\,!n\,!}{(2n)\,!} x^n.$

$$u_n = \frac{(n+1)!n\,!}{(2n)\,!} x^n \quad \Rightarrow \quad u_{n+1} = \frac{(n+2)!(n+1)\,!}{(2n+2)\,!} x^{n+1}.$$

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{(n+2)!(n+1)\,!x^{n+1}}{(2n+2)\,!} \times \frac{(2n)\,!}{(n+1)!n\,!x^n}\right|$$

$$= \lim_{n\to\infty}\left|\frac{(n+2)(n+1)x}{(2n+2)(2n+1)}\right| = \lim_{n\to\infty}\left|\frac{\left(1+\frac{3}{n}+\frac{2}{n^2}\right)x}{4+\frac{6}{n}+\frac{2}{n^2}}\right| = \left|\frac{x}{4}\right|.$$

Converges for $\left|\frac{x}{4}\right| < 1 \quad \Rightarrow \quad -4 < x < 4.$

Dirverges for $\left|\frac{x}{4}\right| > 1 \quad \Rightarrow \quad x > 4 \text{ or } x < -4.$

Inconclusive for $\left|\frac{x}{4}\right| = 1$ i.e. $x = \pm 4.$

Blunders

B1 Error in U_{n+1}

B2 Error in evaluating $\frac{U_{n+1}}{U_n}$

B3 Error in evaluating limit

B4 Range incorrectly or not applied

QUESTION 9

Part (a)	**10 marks**	**Att 3**
Part (b)	**20 (5, 5, 5,5) marks**	**Att (2, 2, 2, 2)**
Part (c)	**20 (5, 5, 5, 5) marks**	**Att (2, 2, 2, 2)**

Part (a) **10 marks** **Att 3**

9 (a) z is a random variable with standard normal distribution.
Find the value of z_1 for which $P(z > z_1) = 0{\cdot}0808$.

Part (a) **10 marks** **Att 3**

9 (a)

$$P(z > z_1) = 0.0808 \;\Rightarrow\; 1 - P(z < z_1) = 0.0808.$$
$$\therefore\; P(z < z_1) = 0.9192 \;\Rightarrow\; z_1 = 1.4.$$

Blunders
B1 Incorrect reading of tables or incorrect area each time

Part (b) **20 (5, 5, 5, 5) marks** **Att (2, 2, 2, 2)**

9 (b) A bag contains the following cardboard shapes:
10 red squares, 15 green squares, 8 red triangles and 12 green triangles.
One of the shapes is drawn at random from the bag.
E is the event that a square is drawn.
F is the event that a green shape is drawn.
(i) Find $P(E \cap F)$.
(ii) Find $P(E \cup F)$.
(iii) State whether E and F are independent events, giving a reason for your answer.
(iv) State whether E and F are mutually exclusive events, giving a reason for your answer.

Part (b) (i) **5 marks** **Att 2**

9 (b) (i) $P(E \cap F) = \dfrac{15}{45}$.

Blunders
B1 Incorrect total possible
B2 Incorrect total favourable
B3 No fraction

Part (b) (ii) **5 marks** **Att 2**

9 (b) (ii) $P(E \cup F) = P(E) + P(F) - P(E \cap F) = \dfrac{25}{45} + \dfrac{27}{45} - \dfrac{15}{45} = \dfrac{37}{45}$.

Blunders
B1 Incorrect total possible
B2 Incorrect total favourable
B3 No fraction

Part (b) (iii)	**5 marks**	**Att 2**

9 (b) (iii)	$P(E).P(F)=\frac{25}{45}.\frac{27}{45}=\frac{15}{45}=P(E\cap F)$. $\therefore$ Independent events.

Blunders
B1 reason not given

Part (b) (iv)	**5 marks**	**Att 2**

9 (b) (iv)	$P(E\cap F)\neq 0$ hence not mutually exclusive

Part (c)	**20 (5,5, 5,5) marks**	**Att (2,2,2,2)**

9 (c)	The marks awarded in an examination are normally distributed with a mean mark of 60 and a standard deviation of 10. A sample of 50 students has a mean mark of 63. Test, at the 5% level of significance, the hypothesis that this is a random sample from the population.

Part (c)	**20 (5,5, 5,5) marks**	**Att (2,2,2,2)**

9 (c)

$$\bar{x}=60,\ \sigma=10 \quad \Rightarrow \quad \sigma_{\bar{x}}=\frac{\sigma}{\sqrt{n}}=\frac{10}{\sqrt{50}}=1.4142.$$

$$\frac{x-\bar{x}}{\sigma_{\bar{x}}}=\frac{63-60}{1.4142}=2.1213>1.96.\ \therefore \text{Not a random sample.}$$

Blunders
B1 Error in standard error of mean
B2 Error in finding Z value
B3 Error in confidence interval

Slips
S1 Arithmetic errors

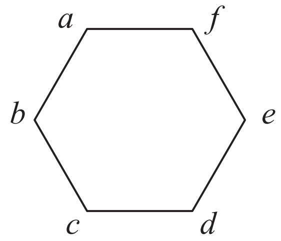
QUESTION 10

Part (a)	**30 (10, 5, 10, 5) marks**	**Att (3, 2, 3, 2)**
Part (b)	**20 (10, 10) marks**	**Att (3, 3)**

Part (a) **30 (10, 5, 10,5) marks** **Att (3, 2, 3,2)**

10 (a) G is the set of rotations that map a regular hexagon onto itself.
$(G, \circ)$ is a group, where $\circ$ denotes composition.

The anti-clockwise rotation through 60^{o} is written as R_{60°.



(i) List the elements of G.
(ii) State which elements of the group, if any, are generators.
(iii) List all the proper subgroups of $(G, \circ)$.
(iv) Find $Z(G)$, the centre of $(G, \circ)$. Justify your answer.

Part (a) (i) **10 marks** **Att 3**

10 (a) (i) $G = \{R_{0^0}, R_{60^0}, R_{120^0}, R_{180^9}, R_{240^0}, R_{300^0}\}$.

Blunders
B1 Each one omitted or incorrect

Part (a) (ii) **5 marks** **Att 2**

10 (a) (ii) Generators are R_{60^0} and R_{300^0}.

Blunders
B1 Each one omitted

Part (a) (iii) **10 marks** **Att 3**

10 (a) (iii) Proper subgroups of $(G, \circ)$ are $\{R_{0^0}, R_{180^0}\}$, and $\{R_{0^0}, R_{120^0}, R_{240^0}\}$.

Blunders
B1 Each one omitted or incorrect

Part (a) (iv) **5 marks** **Att 2**

10 (a) (iv) Each element of G commutes with each of the elements of G.
$\therefore \; Z(G) = G$.

Blunders
B1 Z(G) not found
B3 Error in justification

Part (b) **20 (10, 10) marks** **Att (3, 3)**

10 (b) (i) Show that the group $\left\{\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}, \begin{pmatrix}-1 & 0\\0 & -1\end{pmatrix}\right\}$ under matrix multiplication is isomorphic to the group {0, 1} under addition modulo 2.

(ii) Prove that any infinite cyclic group is isomorphic to $(\mathbf{Z}, +)$.

Part (b) (i) **10 marks** **Att 3**

10 (b) (i) Let $I = \begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}$ and $A = \begin{pmatrix}-1 & 0\\0 & -1\end{pmatrix}$. Let $G = \{I, A\}$ under matrix multiplication.

$$A^2 = \begin{pmatrix}-1 & 0\\0 & -1\end{pmatrix}\begin{pmatrix}-1 & 0\\0 & -1\end{pmatrix} = \begin{pmatrix}1 & 0\\0 & 1\end{pmatrix} = I$$

Thus I is of order 1 and A is of order 2.

Let $H = \{0, 1\}$ under addition modulo 2.

$1+1=2=0 \mod(2)$.

Thus 0 is of order 1 and 1 is of order 2.

As G and H both have one element of order 1 and one element of order 2, they are isomorphic. The isomorphism $G \to H$ is

$I \to 0$

$A \to 1$.

Blunders

B1 Any property of isomorphism not included

Attempts

A1 Elements of H found

Part (b) (ii) **10 marks** **Att 3**

10 (b) (ii) Let $G = <g>$ be any infinite cyclic group generated by g under $*$.

Define $\phi : (G, *) \to (Z, +) : g^k \to k$.

$$\phi\left(g^a * g^b\right) = \phi\left(g^{a+b}\right) = a + b$$
$$= \phi\left(g^a\right) + \phi\left(g^b\right).$$

$\therefore \phi$ is an isomorphism $\Rightarrow (G, *)$ and $(Z, +)$ are isomorphic.

Blunders

B1 Error with indices

B2 No conclusion

QUESTION 11

Part (a)	**10 (5, 5) marks**	**Att (2, 2)**
Part (b)	**20 (10, 10) marks**	**Att (3, 3)**
Part (c)	**20 marks**	**Att 6**

Part (a) **10 (5, 5) marks** **Att (2, 2)**

11 (a) **(i)** Find the image of $a(-1, 2)$ and $b(0, 4)$ under the transformation

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 2 \\ -4 \end{pmatrix}.$$

(ii) Show that ab is parallel to $a'b'$.

Part (a) (i) **5 marks** **Att 2**

11 (a) (i)

$$\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} -1 \\ 2 \end{pmatrix} + \begin{pmatrix} 2 \\ -4 \end{pmatrix} = \begin{pmatrix} -2 \\ 4 \end{pmatrix} + \begin{pmatrix} 2 \\ -4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

$$\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} 0 \\ 4 \end{pmatrix} + \begin{pmatrix} 2 \\ -4 \end{pmatrix} = \begin{pmatrix} 0 \\ 8 \end{pmatrix} + \begin{pmatrix} 2 \\ -4 \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \end{pmatrix}.$$

Blunders

B1 Error in matrix multiplication

B2 Error in addition

Part (a) (ii) **5 marks** **Att 2**

11 (a) (ii) Slope $ab = \dfrac{4-2}{0+1} = 2$ and slope $a'b' = \dfrac{4-0}{2-0} = 2$. $\therefore$ ab is parallel to $a'b'$.

Blunders

B1 Error in slope formula

B2 No conclusion

Part (b) **20 (10, 10) marks** **Att (3, 3)**

11 (b) $p\,(x, y)$ is a point such that the distance from p to the point $(2, 0)$ is half the distance from p to the line $x = 8$.

(i) Find the equation of the locus of p.

(ii) Show that this locus is an ellipse centred at the origin, by expressing its equation in the form $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$.

Part (b) (i) **10 marks** **Att 3**

11 (b) (i)

$p(x, y)$, $s(2, 0)$. $\therefore$ $|ps| = \dfrac{1}{2}\sqrt{(x-2)^2 + (y-0)^2}$.

Distance from p$(x,\ y)$ to line $x = 8$ is $\left|\dfrac{8-x}{1}\right|$.

$\therefore$ Locus of p : $3x^2 + 4y^2 = 48$.

Blunders

B1 Error in each distance formula

Part (b) (ii) **10 marks** **Att 3**

11 (b) (ii)

$$3x^2 + 4y^2 = 48$$

$$\Rightarrow \frac{3x^2}{48} + \frac{4y^2}{48} = 1$$

$$\Rightarrow \frac{x^2}{16} + \frac{y^2}{12} = 1$$

Blunders

B1 Error in format of ellipse equation

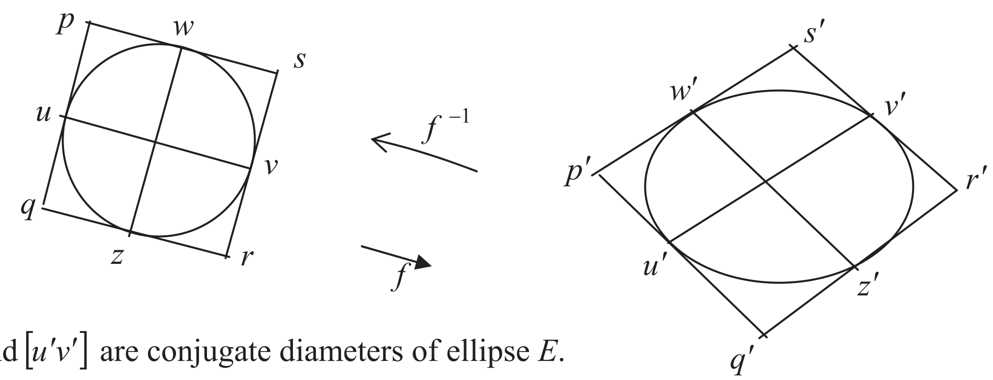
B2 Error in squaring

Part (c) **20 marks** **Att 6**

11 (c) Prove that the areas of all parallelograms circumscribed about a given ellipse at the endpoints of conjugate diameters are equal.

Part (c) **20 marks** **Att 6**

11 (c)



$[w'z']$ and $[u'v']$ are conjugate diameters of ellipse E.

Tangents at their end-points form the parallelogram $p'q'r's'$.

Under an affine transformation f^{-1}, the ellipse maps to the circle $x^2 + y^2 = 1$ and $p'q'r's'$ is mapped to *pqrs*..

$[uv]$ and $[wz]$ are conjugate diameters of the circle and $uv \perp wz$.

The square *pqrs* has fixed area 4 sq units.

Area *pqrs*=2 *area pqr* $\Rightarrow$ area $p'q'r's' = 2$ area $p'q'r'$ as ratio is an invariant map.

Area $p'q'r's' = 2|\text{det.}\, f|\text{area}\ \Delta pqr$

$= |\text{det.}\, f|\text{area}\ pqrs.$

But det. f is constant and area *pqrs* is also constant $\Rightarrow$ area $p'q'r's'$ is constant.

$\therefore$ Areas of all parallelograms at end points of conjugate diameters are equal.

Blunders

B1 Error in mapping

B2 No statement regarding constant area of square

B3 No statement of ratio being invariant and det f being constant

B4 No conclusion

BONUS MARKS FOR ANSWERING THROUGH IRISH

Bonus marks are applied separately to each paper as follows:

If the mark achieved is less than 226, the bonus is 5% of the mark obtained, rounding *down*.
(e.g. 198 marks × 5% = 9.9 ⇒ bonus = 9 marks.)

If the mark awarded is 226 or above, the following table applies:

Marks obtained	Bonus
226 – 231	11
232 – 238	10
239 – 245	9
246 – 251	8
252 – 258	7
259 – 265	6
266 – 271	5
272 – 278	4
279 – 285	3
286 – 291	2
292 – 298	1
299 – 300	0