Coimisiún na Scrúduithe Stáit
State Examinations Commission

Leaving Certificate 2016

Marking Scheme

Applied Mathematics

Higher Level

Note to teachers and students on the use of published marking schemes

Marking schemes published by the State Examinations Commission are not intended to be standalone documents. They are an essential resource for examiners who receive training in the correct interpretation and application of the scheme. This training involves, among other things, marking samples of student work and discussing the marks awarded, so as to clarify the correct application of the scheme. The work of examiners is subsequently monitored by Advising Examiners to ensure consistent and accurate application of the marking scheme. This process is overseen by the Chief Examiner, usually assisted by a Chief Advising Examiner. The Chief Examiner is the final authority regarding whether or not the marking scheme has been correctly applied to any piece of candidate work.

Marking schemes are working documents. While a draft marking scheme is prepared in advance of the examination, the scheme is not finalised until examiners have applied it to candidates' work and the feedback from all examiners has been collated and considered in light of the full range of responses of candidates, the overall level of difficulty of the examination and the need to maintain consistency in standards from year to year. This published document contains the finalised scheme, as it was applied to all candidates' work.

In the case of marking schemes that include model solutions or answers, it should be noted that these are not intended to be exhaustive. Variations and alternatives may also be acceptable. Examiners must consider all answers on their merits, and will have consulted with their Advising Examiners when in doubt.

Future Marking Schemes

Assumptions about future marking schemes on the basis of past schemes should be avoided. While the underlying assessment principles remain the same, the details of the marking of a particular type of question may change in the context of the contribution of that question to the overall examination in a given year. The Chief Examiner in any given year has the responsibility to determine how best to ensure the fair and accurate assessment of candidates' work and to ensure consistency in the standard of the assessment from year to year. Accordingly, aspects of the structure, detail and application of the marking scheme for a particular examination are subject to change from one year to the next without notice.

General Guidelines

1 Penalties of three types are applied to candidates' work as follows:

Slips	- numerical slips	S(-1)
Blunders	- mathematical errors	B(-3)
Misreading	- if not serious	M(-1)

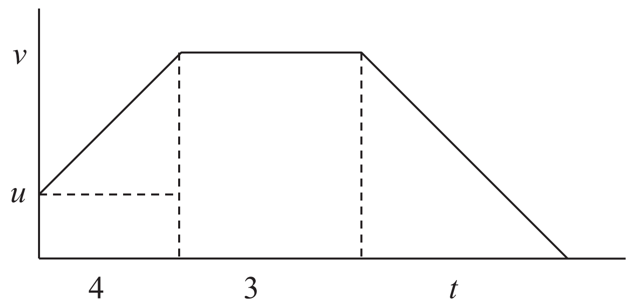
Serious blunder or omission or misreading which oversimplifies:
- award the attempt mark only.

Attempt marks are awarded as follows: 5 (att 2)

2 The marking scheme shows one correct solution to each question.
In many cases there are other equally valid methods.

1. **(a)** A car has an initial speed of u m s^{-1}. It moves in a straight line with constant acceleration f for 4 seconds. It travels 40 m while accelerating.
The car then moves with uniform speed and travels 45 m in 3 seconds.
It is then brought to rest by a constant retardation $2f$.
(i) Draw a speed-time graph for the motion.
(ii) Find the value of u.
(iii) Find the total distance travelled.

(i)



5

(ii)

$$3v = 45$$

$$v = 15$$ 5

$$4u + \tfrac{1}{2}(4)(15 - u) = 40$$

$$4u + 30 - 2u = 40$$

$$u = 5 \text{ m s}^{-1}$$ 5

(iii)

$$f = \frac{15 - 5}{4} = 2.5$$ 5

$$t = \frac{15}{2f} = 3$$

$$d = 40 + 45 + \tfrac{1}{2}(3)15$$

$$= 107.5 \text{ m}$$ 5

25

1. (b) A particle is projected vertically upwards with a velocity of u m s^{-1}.
After an interval of $2t$ seconds a second particle is projected vertically upwards from the same point and with the same initial velocity.

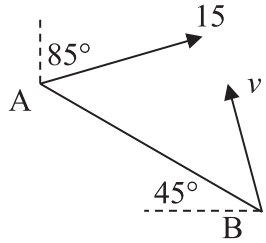
They meet at a height of h m.

Show that $h = \dfrac{u^2 - g^2t^2}{2g}$.

$s_1 = ut_1 - \frac{1}{2}g(t_1)^2$	5
$s_2 = u(t_1 - 2t) - \frac{1}{2}g(t_1 - 2t)^2$	5
$s_1 = s_2$	
$ut_1 - \frac{1}{2}g(t_1)^2 = u(t_1 - 2t) - \frac{1}{2}g(t_1 - 2t)^2$	5
$ut_1 - \frac{1}{2}g(t_1)^2 = ut_1 - 2ut - \frac{1}{2}g(t_1)^2 + 2gt_1t - 2gt^2$	
$\Rightarrow t_1 = \dfrac{u}{g} + t$	5
$h = ut_1 - \frac{1}{2}g(t_1)^2$	
$h = u\left(\dfrac{u}{g} + t\right) - \frac{1}{2}g\left(\dfrac{u}{g} + t\right)^2$	
$= \dfrac{u^2}{2g} - \dfrac{gt^2}{2}$	
$h = \dfrac{u^2 - g^2t^2}{2g}$	5

25

2 **(a)** At 12 noon, ship A is north west of ship B as shown.



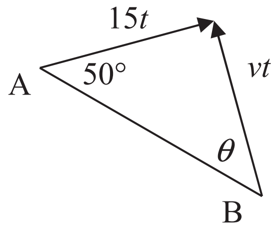
Ship A is moving north 85° east at a uniform speed of 15 km h^{-1}.

Ship B is moving in a straight line with uniform speed v km h^{-1}.

Ship B intercepts ship A.

(i) Find the least possible value of v.

(ii) If $v = 13$ km h^{-1}, find the two possible directions that ship B can travel in order to intercept ship A.



(i)

$$\frac{15t}{\sin\theta} = \frac{vt}{\sin 50}$$ 5

$$v = \frac{15\sin 50}{\sin\theta}$$ 5

$$v_{\text{min}} \Rightarrow \sin\theta = 1$$ 5

$$v_{\text{min}} = 15\sin 50$$

$$= 11.49 \text{ km h}^{-1}$$ 5

(ii)

$$\frac{15t}{\sin\alpha} = \frac{13t}{\sin 50}$$

$$\sin\alpha = \frac{15\sin 50}{13}$$

$$= 0.8839$$

$$\alpha = 62.1° \text{ or } 117.9°$$ 5

25

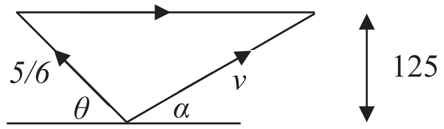
2 (b) A man can swim at $\frac{5}{6}$ m s^{-1} in still water. He swims across a river 125 m wide.

The river flows at a constant speed of $\frac{25}{18}$ m s^{-1} parallel to the straight banks.

How long will it take him if he swims so as to reach the opposite bank

(i) as quickly as possible

(ii) as little downstream as possible?



(i) $$t = \frac{125}{\frac{5}{6}\sin\theta}$$ 5

$$\sin\theta = 1$$ 5

$$t = \frac{125}{\frac{5}{6}}$$

$$= 150 \text{ s}$$ 5

(ii) $$v\sin\alpha = \tfrac{5}{6}\sin\theta$$

$$v\cos\alpha = \tfrac{25}{18} - \tfrac{5}{6}\cos\theta$$

$$\tan\alpha = \frac{\sin\theta}{\frac{5}{3} - \cos\theta}$$ 5

$$\frac{d(\tan\alpha)}{d\theta} = \frac{(\frac{5}{3} - \cos\theta)\cos\theta - (\sin\theta)\sin\theta}{(\frac{5}{3} - \cos\theta)^2}$$

$$= 0$$

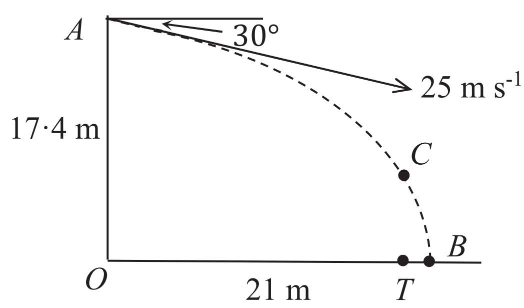
$$\Rightarrow \tfrac{5}{3}\cos\theta - \cos^2\theta - \sin^2\theta = 0$$

$$\Rightarrow \cos\theta = \tfrac{3}{5}, \quad \sin\theta = \tfrac{4}{5}$$

$$t = \frac{125}{\frac{5}{6}\sin\theta}$$

$$= 187.5 \text{ s}$$ 5 | 25

3. **(a)** A ball is thrown from a point A at a target T, which is on horizontal ground. The point A is 17·4 m vertically above the point O on the ground. The ball is thrown from A with speed 25 m s^{-1} at an angle of 30° below the horizontal. The distance OT is 21 m.
The ball misses the target and hits the ground at the point B, as shown in the diagram.



Find
(i) the time taken for the ball to travel from A to B
(ii) the distance TB.

The point C is on the path of the ball vertically above T.
(iii) Find the speed of the ball at C.

(i)

$$17.4 = 25\sin 30.t + \tfrac{1}{2}gt^2$$ 5

$$17.4 = 12.5t + 4.9t^2$$

$$t = 1$$ 5

(ii)

$$|OB| = 25\cos 30.(1) = 21.65$$

$$|TB| = 21.65 - 21$$

$$= 0.65 \text{ m}$$ 5

(iii)

$$25\cos 30 \times t = 21$$

$$t = 0.97$$ 5

$$v_{\vec{i}} = 25\cos 30 = 21.65$$

$$v_{\vec{j}} = 25\sin 30 + g(0.97) = 22.01$$

$$v = \sqrt{21.65^2 + 22.01^2}$$

$$= 30.87 \text{ m s}^{-1}$$ 5

25

3 (b) A plane is inclined at an angle of 60° to the horizontal.
A particle is projected up the plane with initial speed u m s^{-1} at an angle θ to the inclined plane. The plane of projection is vertical and contains the line of greatest slope.

The maximum range of the particle is $\dfrac{ku^2}{g}$.

Find the value of k correct to one decimal place.

$$r_j = 0$$

$$u\sin\theta \times t - \tfrac{1}{2} g\cos 60 \times t^2 = 0 \qquad 5$$

$$t = \frac{2u\sin\theta}{g\cos 60} = \frac{4u\sin\theta}{g} \qquad 5$$

$$r_{\vec{i}} = u\cos\theta \times t - \tfrac{1}{2} g\sin 60 \times t^2$$

$$R = u\cos\theta \times \frac{4u\sin\theta}{g} - \tfrac{1}{2} g\sin 60 \times \left(\frac{4u\sin\theta}{g}\right)^2 \qquad 5$$

$$R = \frac{2u^2\sin 2\theta}{g} - \frac{4\sqrt{3}u^2\sin^2\theta}{g}$$

$$\frac{dR}{d\theta} = \frac{4u^2\cos 2\theta}{g} - \frac{8\sqrt{3}u^2\sin\theta\cos\theta}{g}$$

$$= 0$$

$$\frac{4u^2\cos 2\theta}{g} = \frac{8\sqrt{3}u^2\sin\theta\cos\theta}{g}$$

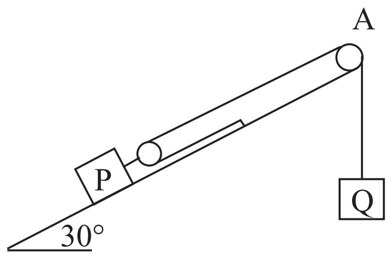
$$\frac{4u^2\cos 2\theta}{g} = \frac{4\sqrt{3}u^2\sin 2\theta}{g}$$

$$\tan 2\theta = \frac{1}{\sqrt{3}} \qquad \Rightarrow \theta = 15° \qquad 5$$

$$R = \frac{u^2}{g} - \frac{0.46u^2}{g} = \frac{0.54u^2}{g}$$

$$\Rightarrow k = 0.5 \qquad 5 \qquad 25$$

4. **(a)** The block P has a light pulley fixed to it. The two blocks P and Q, of mass 40 kg and 30 kg respectively, are connected by a taut light inextensible string passing over a light smooth fixed pulley, A, as shown in the diagram.



P is on a rough plane which is inclined at 30° to the horizontal. The coefficient of friction between P and the inclined plane is $\frac{1}{4}$.

Q is hanging freely. The system is released from rest.

Find

(i) the acceleration of P and the acceleration of Q

(ii) the speed of P when it has moved 30 cm.

(i) $30g - T = 30(2a)$ 5

$2T - 40g\sin 30 - \frac{1}{4}(40g\cos 30) = 40a$ 5

$60g - 20g - 5g\sqrt{3} = 160a$

$a = 1.92$

$a_P = 1.92 \quad a_Q = 3.84 \text{ m s}^{-2}$ 5

(ii) $v^2 = u^2 + 2as$

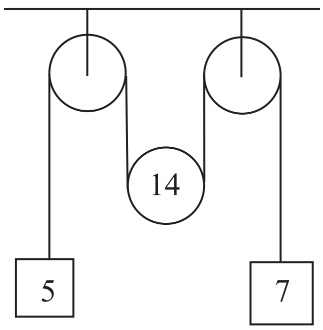
$= 0 + 2(1.92)(0.3)$

$v^2 = 1.152$

$v = 1.07 \text{ m s}^{-1}$ 5

20

4. (b) A light inextensible string passes over a small smooth fixed pulley, under a small smooth moveable pulley, of mass 14 kg, and then over a second small smooth fixed pulley. A 5 kg mass is attached to one end of the string and a 7 kg mass is attached to the other end.



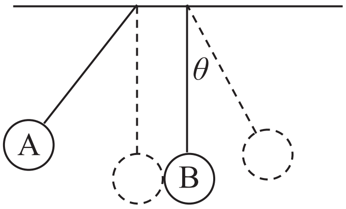
The system is released from rest.

(i) Find the tension in the string.

(ii) If instead of the system starting from rest, the moveable pulley is given an initial upward velocity of 0·8 m s^{-1}, find the time taken until the moveable pulley reverses direction.

(i)	$T - 5g = 5a$	5	
	$T - 7g = 7b$	5	
	$14g - 2T = 14\left(\frac{a+b}{2}\right)$	5	
	$= 7a + 7b$		
	$14g - 2T = 7\left(\frac{T}{5} - g\right) + 7\left(\frac{T}{7} - g\right)$		
	$T = \frac{70g}{11} = 62.36 \text{ N}$	5	
(ii)	$\frac{a+b}{2} = \frac{14g - 2T}{14}$		
	$= -\frac{g}{11} = -0.89$	5	
	$v = u + at$		
	$0 = 0.8 - 0.89t$		
	$t = 0.90 \text{ s}$	5	30

5. **(a)** Two small smooth spheres A, of mass 2 kg, and B, of mass 3 kg, are suspended by light strings from a ceiling as show in the diagram. The distance from the ceiling to the centre of each sphere is 2 m.



Sphere A is drawn back 60° and released from rest. A collides with B and rebounds. B swings through an angle θ.

The coefficient of restitution between the spheres is $\frac{3}{4}$.

(i) Show that A strikes B with a speed of $\sqrt{2g}$ m s^{-1}.

(ii) Find the speed of each sphere after the collision.

(iii) Find the value of θ.

(i)

$$\tfrac{1}{2}(2)u^2 = 2g(2 - 2\cos 60)$$

$$u = \sqrt{2g}$$ 5

(ii) PCM $$2\sqrt{2g} + 3(0) = 2v_1 + 3v_2$$ 5

NEL $$v_1 - v_2 = -\frac{3}{4}\left(\sqrt{2g} - 0\right)$$ 5

$$v_1 = -\frac{\sqrt{2g}}{20} = -0.22 \text{ m s}^{-1}$$

$$v_2 = \frac{7\sqrt{2g}}{10} = 3.10 \text{ m s}^{-1}$$ 5

(iii)

$$\tfrac{1}{2}\times 3 \times 3.10^2 = 3g(2 - 2\cos\theta)$$

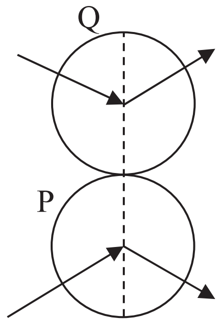
$$\cos\theta = 0.7548$$

$$\theta = 40.99°$$ 5 25

5. (b) Two identical smooth spheres P and Q collide.

The velocity of P **after** impact is $3\vec{i} - \vec{j}$ and the velocity of Q **after** impact is $2\vec{i} + \vec{j}$, where $\vec{j}$ is along the line of the centres of the spheres at impact.

The coefficient of restitution between the spheres is $\frac{1}{2}$.



Find

(i) the velocities, in terms of $\vec{i}$ and $\vec{j}$, of the two spheres before impact

(ii) to the nearest degree, the angle through which the direction of motion of P is deflected by the collision.

P	m	$3\vec{i} + u_1\vec{j}$	$3\vec{i} - \vec{j}$
Q	m	$2\vec{i} + u_2\vec{j}$	$2\vec{i} + \vec{j}$

(i) PCM $\quad m(u_1) + m(u_2) = m(-1) + m(1)$ 5

NEL $\quad -1 - 1 = -\frac{1}{2}(u_1 - u_2)$ 5

$$u_1 = 2 \qquad u_2 = -2$$

$$\vec{v}_P = 3\vec{i} + 2\vec{j}$$
$$\vec{v}_Q = 2\vec{i} - 2\vec{j}$$ 5

(ii) $\tan\alpha = \frac{2}{3} \Rightarrow \alpha = 33.69°$ 5

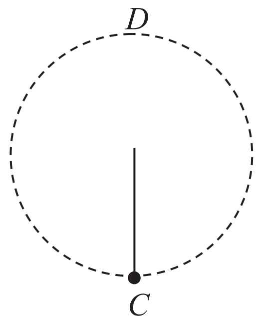
$\tan\beta = \frac{-1}{3} \Rightarrow \beta = -18.43°$

$\theta = 33.69 + 18.43 = 52°$ 5 **25**

6. **(a)** A small particle hanging on the end of a light inextensible string 2 m long is projected horizontally from the point *C*.

(i) Calculate the least speed of projection needed to ensure that the particle reaches the point *D* which is vertically above *C*.

(ii) If the speed of projection is 7 m s^{-1} find the angle that the string makes with the vertical when it goes slack.



(i)
$$\tfrac{1}{2}mu^2 = \tfrac{1}{2}mv^2 + mg(4)$$ 5
$$v^2 = u^2 - 8g$$

$$T + mg = \frac{mv^2}{r}$$
$$0 + mg = \frac{mv^2}{2}$$
$$v^2 = 2g$$
$$\Rightarrow u = \sqrt{10g} = 7\sqrt{2} \text{ m s}^{-1}$$ 5

(ii)
$$\tfrac{1}{2}m(7)^2 = \tfrac{1}{2}mv^2 + mg(2 + 2\cos\alpha)$$ 5
$$v^2 = 49 - 4g - 4g\cos\alpha$$
$$= g - 4g\cos\alpha$$
$$T + mg\cos\alpha = \frac{mv^2}{r}$$
$$0 + mg\cos\alpha = \frac{mv^2}{2}$$ 5
$$v^2 = 2g\cos\alpha$$
$$g - 4g\cos\alpha = 2g\cos\alpha$$
$$\cos\alpha = \frac{1}{6}$$
$$\alpha = 80.41°$$ 5

25

6. (b) A particle P of mass 2 kg is hanging from one end of a light elastic string, of natural length 1 m and elastic constant 98 N m^{-1}. The other end of the string is attached to a fixed point *A*.

The particle is now pulled down to a point *Q* which is 0·4 m vertically below the equilibrium position and released from rest.

(i) Prove that, while the string is taut, P moves with simple harmonic motion.

(ii) Find the speed of P when the string first becomes slack (no longer taut).

(iii) Find the time taken, from release, for P to reach the highest point in its motion.

(i)	$T = 2g$	
	$98e = 2g \quad \Rightarrow \quad e = 0.2$	5
	$2a = 2g - 98(0.2 + x)$	
	$a = -49x$	
	$\Rightarrow$ S.H.M. $(\omega = 7,\ \text{A} = 0.4)$	5
(ii)	$v^2 = \omega^2\left(A^2 - x^2\right)$	
	$= 49\left(0.4^2 - (-0.2)^2\right) = 5.88$	
	$\Rightarrow v = 2.42 \text{ m s}^{-1}$	5
(iii)	$x = A\sin\omega t_1$	
	$0.2 = 0.4\sin 7t_1$	
	$t_1 = \dfrac{\pi}{42} = 0.0748$	5
	$v = u + at$	
	$0 = 2.42 - 9.8t_2$	
	$t_2 = 0.2469$	
	$t = \dfrac{T}{4} + t_1 + t_2$	
	$= \dfrac{2\pi}{4\omega} + 0.0748 + 0.2469$	
	$= 0.5461 \text{ s}$	5

25

7. **(a)** A uniform beam *AB* of length 30 m and mass 200 kg is held in limiting equilibrium by a light inextensible cable attached to *B* as shown in the diagram.

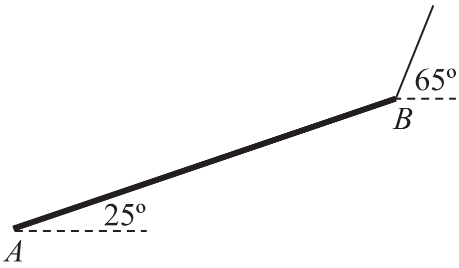
End A of the beam rests on a smooth horizontal surface.

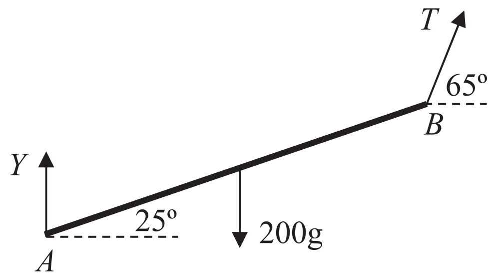
The angle between the beam and the surface is 25º and the cable makes an angle of 65º with the horizontal.

Find

(i) the tension in the cable

(ii) the magnitude of the reaction at *A*.





(i) $T\sin 40 \times 30 = 200g \times 15\cos 25$ **5**

$T = 1381.765\text{ N}$ **5**

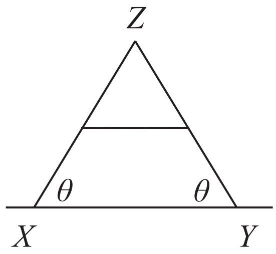
(ii) $T\sin 65 + Y = 200g$ **5**

$Y = 200g - 1381.765\sin 65$

$= 707.6956$ **5** **20**

Note: The question should have stated that the beam rested on a rough horizontal surface. Award (20) marks to candidates whose approach was based on the beam being on either a smooth or a rough surface.

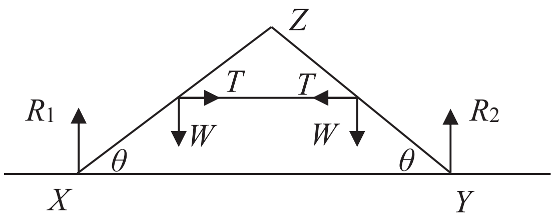
7. (b) Two uniform rods, *XZ* and *YZ*, of length 2 m and weight *W*, are freely jointed at *Z*, and rest in equilibrium in a vertical plane with the ends *X* and *Y* on a smooth horizontal plane. Each rod is inclined at an angle θ to the horizontal. A string connects the mid points of the rods.



(i) Show that the tension in the string is $\dfrac{W}{\tan\theta}$.

A weight $2W$ is placed 25 cm from *X* on *XZ*.

(ii) Show that the tension of the string is increased by 25%.



(i)

$$R_2(4\cos\theta)+T(\sin\theta)= W(3\cos\theta)+W(\cos\theta)+T(\sin\theta)$$

$$R_2 = W \qquad 5$$

$$R_1+R_2=2W \quad \Rightarrow \quad R_1=W \qquad 5$$

$$R_1(2\cos\theta)=W(\cos\theta)+T(\sin\theta) \qquad 5$$

$$2W = W + T\tan\theta$$

$$T=\frac{W}{\tan\theta} \qquad 5$$

(ii)

$$R_2(4)=W(3)+W(1)+2W\left(\tfrac{1}{4}\right)$$

$$R_2=\frac{9W}{8}$$

$$R_1+R_2=4W \quad \Rightarrow \quad R_1=\frac{23W}{8} \qquad 5$$

$$R_1(2\cos\theta)=W(\cos\theta)+2W\left(\tfrac{7}{4}\cos\theta\right)+T_1(\sin\theta)$$

$$\tfrac{23}{8}W\times 2 = W+\tfrac{7}{2}W+T_1\tan\theta$$

$$T_1=\frac{5W}{4\tan\theta}$$

$$T_1-T=\frac{W}{4\tan\theta}=25\% \text{ of } T. \qquad 5$$

30

8. **(a)** Prove that the moment of inertia of a uniform rod, of mass m and length 2ℓ, about an axis through its centre, perpendicular to its plane, is $\frac{1}{3}m\ell^2$.

Let M = mass per unit length

$$\text{mass of element} = M\{dx\}$$

$$\text{moment of inertia of the element} = M\{dx\}x^2 \qquad 5$$

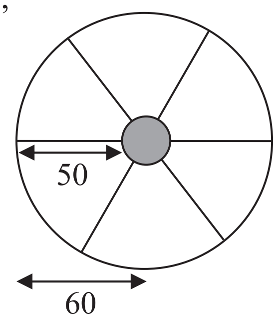
$$\text{moment of inertia of the rod} = M\int_{-\ell}^{\ell} x^2\,dx \qquad 5$$

$$= M\left[\frac{x^3}{3}\right]_{-\ell}^{\ell} \qquad 5$$

$$= \tfrac{2}{3}M\ell^3$$

$$= \tfrac{1}{3}m\ell^2 \qquad 5 \qquad 20$$

8. (b) A wheel, of radius 60 cm, is formed of a thin uniform rim (hoop), six uniform spokes and an axle in the shape of a disc.
The mass of the rim is 4 kg.
Each spoke has a mass of 0·05 kg and length 50 cm.
The mass of the axle is 1 kg and it has a radius of 10 cm.
The wheel is rolling on a horizontal road at a speed of 5 m s^{-1}.



(i) Find the moment of inertia of the wheel about an axis through the centre of the axle, perpendicular to its plane.

(ii) Calculate the kinetic energy of the wheel.

(iii) If the wheel comes to an incline of $\sin^{-1}\frac{1}{5}$ how far will it travel up the incline before it stops?

(i)

$$I = 4\times 0.6^2 + \tfrac{1}{2}\times 1\times 0.1^2$$ 5

$$+\,6\left\{\tfrac{1}{3}\times 0.05\times 0.25^2 + 0.05\times 0.35^2\right\} = 1.488 \text{ kg m}^2$$ 5

(ii)

$$v = r\omega \Rightarrow \omega = \frac{5}{0.6} = \frac{25}{3}$$ 5

$$E_K = \tfrac{1}{2}I\omega^2 + \tfrac{1}{2}mv^2$$

$$= \tfrac{1}{2}(1.488)\left(\tfrac{25}{3}\right)^2 + \tfrac{1}{2}(5.3)(25)$$

$$= 117.917 \text{ J}$$ 5

(iii)

$$5.3gd\sin\phi = 117.917$$ 5

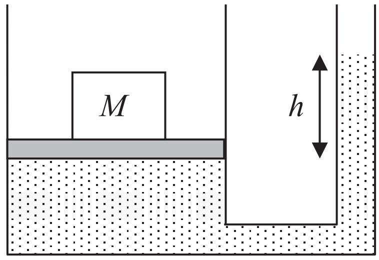
$$d = \frac{117.917}{5.3\times 9.8\times 0.2}$$

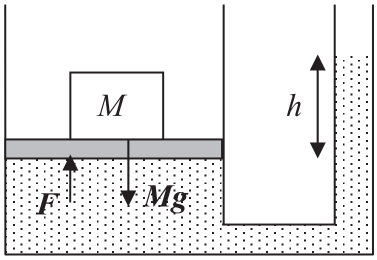
$$= 11.35 \text{ m.}$$ 5

30

9. **(a)** A load of mass *M* acts on a light circular piston of diameter *d*.
The piston sits on a reservoir of oil.
The density of the oil is ρ.
The reservoir is connected to a round tube.
The oil rises in the open tube to a height *h*.

Find *h* in terms of *M*, ρ and *d*.





$$F = P \times A$$ 5

$$= \rho g h \times \frac{\pi d^2}{4}$$ 5

$$F = Mg$$ 5

$$\rho g h \times \frac{\pi d^2}{4} = Mg$$

$$h = \frac{Mg}{\rho g \times \frac{\pi d^2}{4}}$$

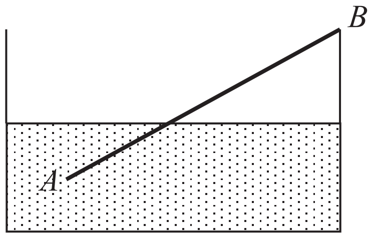
$$= \frac{4M}{\pi \rho d^2}$$ 5

20

9 **(b)** A thin uniform rod AB is in equilibrium in an inclined position in a container of water.

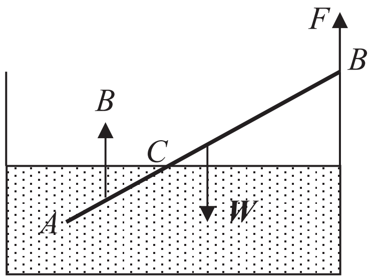
End B is supported by the edge of the container as shown in the diagram.

The relative density of the rod is s.



Find in terms of s the fraction of the length of the rod that is immersed in the water.

[Density of water = 1000 kg m^{-3}]



$$|AB| = \ell$$

$$|AC| = x$$

$$B = \frac{\frac{x}{\ell}W(1)}{s} = \frac{xW}{s\ell} \qquad 5$$

$$B \times \left(\ell - \tfrac{1}{2}x\right)\sin\theta \;=\; W \times \tfrac{1}{2}\ell\sin\theta \qquad 5,5$$

$$\frac{xW}{s\ell} \times \left(\ell - \tfrac{1}{2}x\right)\sin\theta \;=\; W \times \tfrac{1}{2}\ell\sin\theta$$

$$\frac{x}{s\ell} \times \left(\ell - \tfrac{1}{2}x\right) \;=\; \tfrac{1}{2}\ell$$

$$x^2 - 2\ell x + s\ell^2 = 0 \qquad 5$$

$$x = \ell - \ell\sqrt{1-s} \qquad 5$$

$$\frac{x}{\ell} = 1 - \sqrt{1-s} \qquad 5$$

30

10. **(a)** At time t seconds the acceleration a m s^{-2} of a particle, P, is given by

$$a = 8t + 4.$$

At $t = 0$, P passes through a fixed point with velocity -24 m s^{-1}.

(i) Show that P changes its direction of motion only once in the subsequent motion.

(ii) Find the distance travelled by P between $t = 0$ and $t = 3$.

(i)

$$\frac{dv}{dt} = 8t + 4$$

$$v = 4t^2 + 4t + C$$

$$-24 = 0 + 0 + C$$

$$v = 4t^2 + 4t - 24$$ 5

$$v = 0 \Rightarrow t = -3, t = 2$$ 5

$t = 2 \Rightarrow$ changes direction once 5

(ii)

$$\frac{ds}{dt} = 4t^2 + 4t - 24$$

$$S = \frac{4t^3}{3} + 2t^2 - 24t + C_1$$

$$0 = 0 + 0 - 0 + C_1$$

$$S(t) = \frac{4t^3}{3} + 2t^2 - 24t$$ 5

$$S(0) = 0$$

$$S(2) = \frac{32}{3} + 8 - 48 = -29.33$$

$$S(3) = 36 + 18 - 72 = -18$$

$$d = 29.33 + (29.33 - 18)$$

$$= 40.66 \text{ m}$$: 5

25

10 **(b)** A particle moves along a straight line in such a way that its acceleration is always directed towards a fixed point O on the line, and is proportional to its displacement from that point.

The displacement of the particle from O at time t is x.

The equation of motion is

$$v\frac{dv}{dx} = -\omega^2 x$$

where v is the velocity of the particle at time t and ω is a constant.

The particle starts from rest at a point P, a distance A from O.

Derive an expression for

(i) v in terms of A, ω and x

(ii) x in terms of A, ω and t.

(i) $\int v\,dv = -\omega^2 \int x\,dx$ — 5

$\frac{1}{2}v^2 = -\frac{1}{2}\omega^2 x^2 + C$ — 5

$0 = -\frac{1}{2}\omega^2 A^2 + C$

$\frac{1}{2}v^2 = -\frac{1}{2}\omega^2 x^2 + \frac{1}{2}\omega^2 A^2$

$v = \pm\omega\sqrt{A^2 - x^2}$ — 5

(ii) $\frac{dx}{dt} = -\omega\sqrt{A^2 - x^2}$ or $\omega\sqrt{A^2 - x^2}$

$\int \frac{dx}{\sqrt{A^2 - x^2}} = -\int \omega\,dt$

$\sin^{-1}\left(\frac{x}{A}\right) = -\omega t + C_1$ — 5

$x = A\sin(-\omega t + C_1)$

$A = A\sin(0 + C_1)$

$C_1 = \frac{\pi}{2}$

$x = A\sin\left(-\omega t + \frac{\pi}{2}\right)$

$x = A\cos\omega t$ — 5 — **25**