2012. M329

Coimisiún na Scrúduithe Stáit
State Examinations Commission

Leaving Certificate Examination, 2012

Mathematics (Project Maths – Phase 3)

Paper 1

Higher Level

Friday 8 June Afternoon 2:00 – 4:30

300 marks

Examination number

Centre stamp

Running total	

For examiner	
Question	Mark
1	
2	
3	
4	
5	
6	
7	
8	
9	
Total	

Grade

Instructions

There are **two** sections in this examination paper:

Section A	Concepts and Skills	150 marks	6 questions
Section B	Contexts and Applications	150 marks	3 questions

Answer all nine questions.

Write your answers in the spaces provided in this booklet. You will lose marks if you do not do so. There is space for extra work at the back of the booklet. You may also ask the superintendent for more paper. Label any extra work clearly with the question number and part.

The superintendent will give you a copy of the *Formulae and Tables* booklet. You must return it at the end of the examination. You are not allowed to bring your own copy into the examination.

Marks will be lost if all necessary work is not clearly shown.

Answers should include the appropriate units of measurement, where relevant.

Answers should be given in simplest form, where relevant.

Write the make and model of your calculator(s) here:

Section A Concepts and Skills 150 marks

Answer **all six** questions from this section.

Question 1 **(25 marks)**

(a) Solve the simultaneous equations:

$$a^2 - ab + b^2 = 3$$
$$a + 2b + 1 = 0$$

(b) Find the set of all real values of x for which $\frac{2x-5}{x-3} \leq \frac{5}{2}$.

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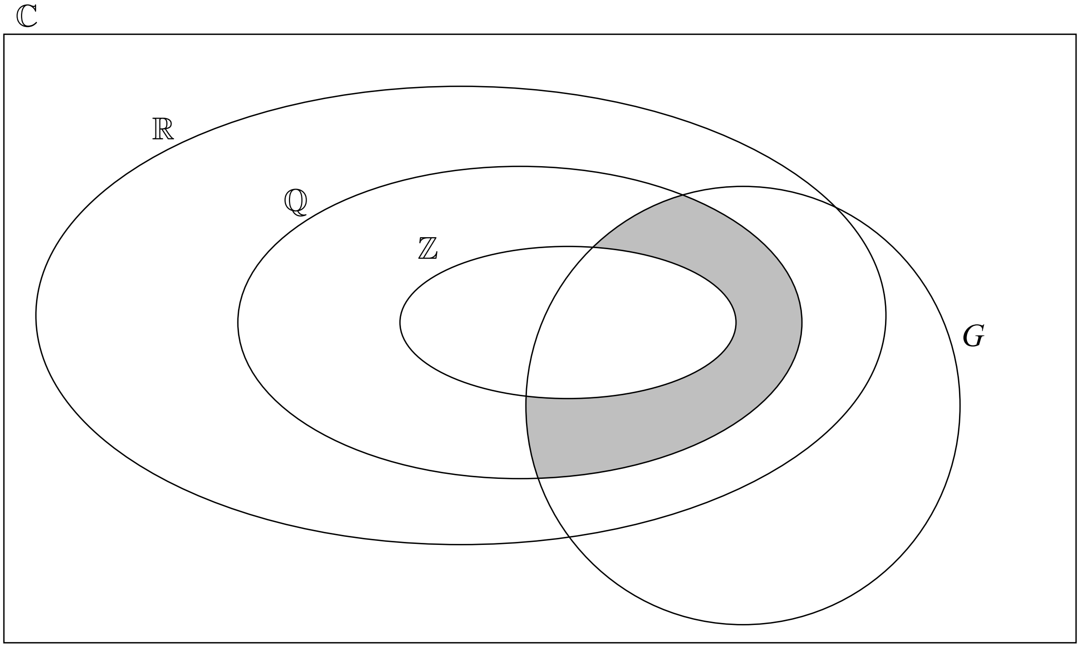
Question 2 **(25 marks)**

Let G be the set $\{x + yi \mid x, y \in \mathbb{Z},\ i^2 = -1\}$.

Consider the Venn diagram below.



(a) There are three regions in the diagram that represent empty sets. One of these is shaded. Shade in the other two.

(b) Insert each of the following numbers in its correct region on the diagram.

$\sqrt{2}$	7	$\sqrt{3} - i$
$4 + 3i$	$\frac{1}{2}$	$\frac{1}{2} + 2i$

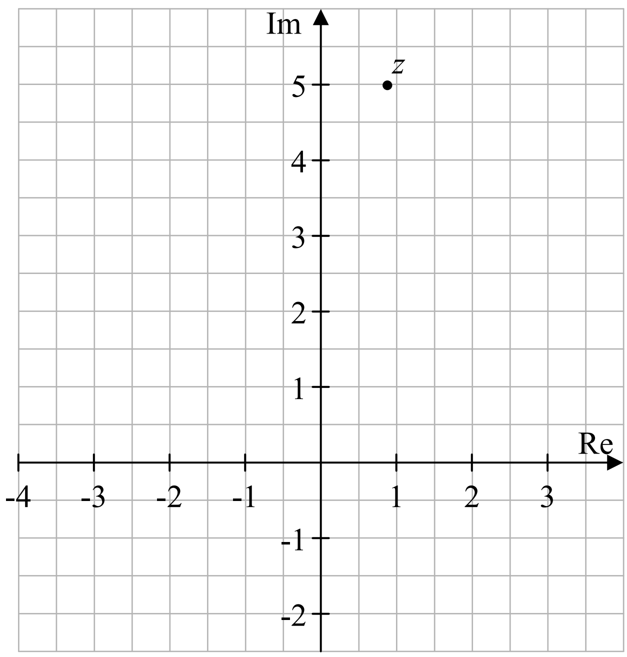
(c) Consider the product ab, where $a \in G$ and $b \in \mathbb{Q}$. There is a non-empty region in the diagram where ab cannot be. Write the word "here" in this region.

Question 3 **(25 marks)**

The complex number z has modulus $5\frac{1}{16}$ and argument $\frac{4\pi}{9}$.

(a) Find, in polar form, the four complex fourth roots of z.
(That is, find the four values of w for which $w^4 = z$.)

(b) z is marked on the Argand diagram below.
On the same diagram, show the four answers to part **(a)**.



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Question 4 **(25 marks)**

(a) Prove, by induction, the formula for the sum of the first n terms of a geometric series. That is, prove that, for $r \neq 1$:

$$a + ar + ar^2 + \cdots + ar^{n-1} = \frac{a(1-r^n)}{1-r}.$$

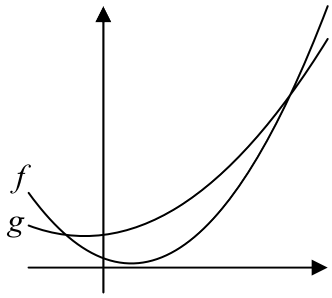
(b) By writing the recurring part as an infinite geometric series, express the following number as a fraction of integers:

$$5{\cdot}\dot{2}\dot{1} = 5{\cdot}2121212121\ldots$$

Question 5 **(25 marks)**

The functions f and g are defined for $x \in \mathbb{R}$ as

$$f : x \mapsto 2x^2 - 3x + 2 \quad \text{and}$$
$$g : x \mapsto x^2 + x + 7 .$$



(a) Find the co-ordinates of the two points where the curves $y = f(x)$ and $y = g(x)$ intersect.

(b) Find the area of the region enclosed between the two curves.

There is space to continue your work on the next page.

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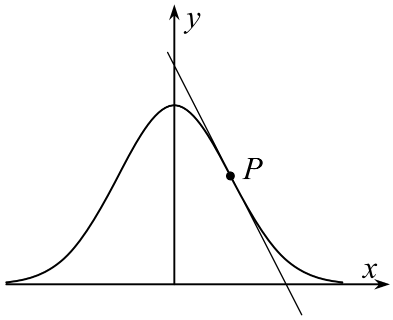
Question 6 **(25 marks)**

(a) Let $f(x)=e^{-\frac{1}{2}x^2}$.

Show that the second derivative of $f(x)$ with respect to x is $f''(x)=\left(x^2-1\right)e^{-\frac{1}{2}x^2}$.

(b) The point P in the first quadrant is a point of inflection of the curve $y = e^{-\frac{1}{2}x^2}$.

Show that the tangent at P crosses the x-axis at $(2,0)$.



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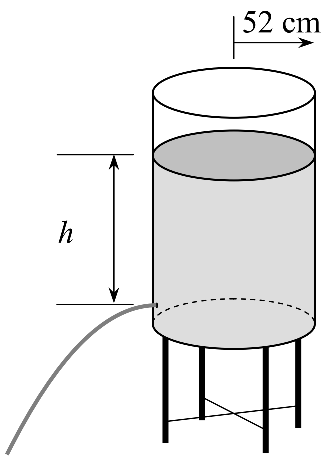
Section B — Contexts and Applications — 150 marks

Answer **all three** questions from this section.

Question 7 **(50 marks)**

An open cylindrical tank of water has a hole near the bottom. The radius of the tank is 52 cm. The hole is a circle of radius 1 cm. The water level gradually drops as water escapes through the hole.



Over a certain 20-minute period, the height of the surface of the water is given by the formula

$$h = \left(10 - \frac{t}{200}\right)^2$$

where h is the height of the surface of the water, in cm, as measured from the centre of the hole,
and t is the time in seconds from a particular instant $t = 0$.

(a) What is the height of the surface at time $t = 0$?

(b) After how many seconds will the height of the surface be 64 cm?

(c) Find the rate at which the **volume** of water in the tank is decreasing at the instant when the height is 64 cm.
Give your answer correct to the nearest cm^3 per second.

(d) The rate at which the volume of water in the tank is decreasing is equal to the speed of the water coming out of the hole, multiplied by the area of the hole. Find the speed at which the water is coming out of the hole at the instant when the height is 64 cm.

(e) Show that, as t varies, the speed of the water coming out of the hole is a constant multiple of $\sqrt{h}$.

(f) The speed, in centimetres per second, of water coming out of a hole like this is known to be given by the formula

$$v = c\sqrt{1962h}$$

where c is a constant that depends on certain features of the hole.
Find, correct to one decimal place, the value of c for this hole.

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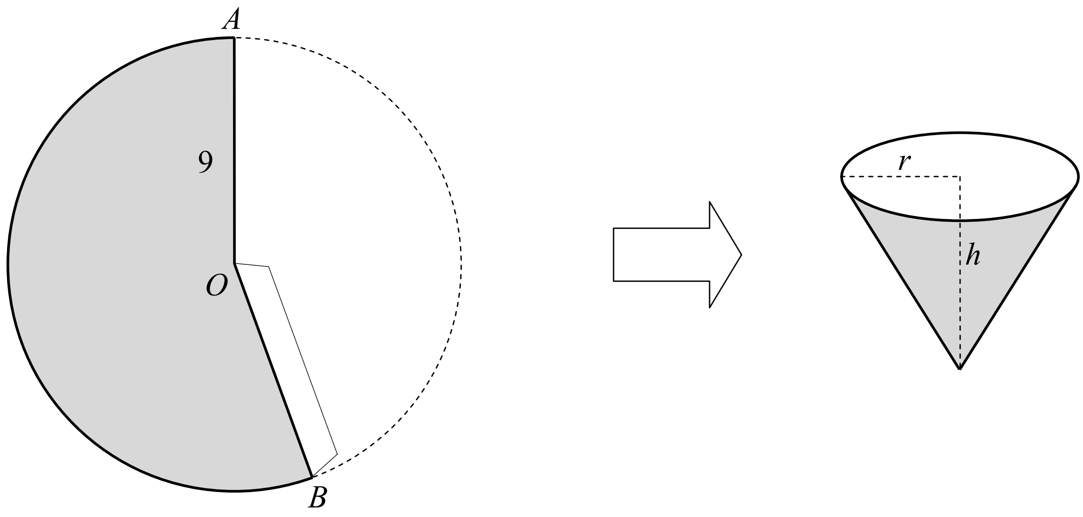
Question 8 **(50 marks)**

A company uses waterproof paper to make disposable conical drinking cups. To make each cup, a sector AOB is cut from a circular piece of paper of radius 9 cm. The edges AO and OB are then joined to form the cup, as shown.

The radius of the rim of the cup is r, and the height of the cup is h.



(a) By expressing r^2 in terms of h, show that the capacity of the cup, in cm^3, is given by the formula

$$V = \frac{\pi}{3} h\left(81 - h^2\right).$$

(b) There are two positive values of h for which the capacity of the cup is $\dfrac{154\pi}{3}$.

One of these values is an integer.
Find the two values.
Give the non-integer value correct to two decimal places.

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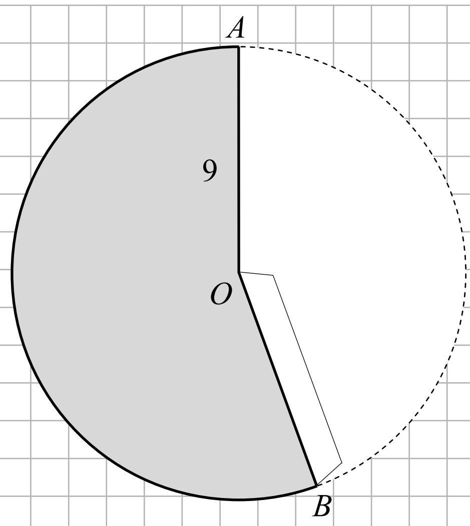
(c) Find the maximum possible volume of the cup, correct to the nearest cm^3.

(d) Complete the table below to show the radius, height, and capacity of each of the cups involved in parts **(b)** and **(c)** above.
In each case, give the radius and height correct to two decimal places.

	cups in part **(b)**		cup in part **(c)**
radius (r)			
height (h)			
capacity (V)	$\frac{154\pi}{3} \approx 161 \text{ cm}^3$	$\frac{154\pi}{3} \approx 161 \text{ cm}^3$	

(e) In practice, which one of the three cups above is the most reasonable shape for a conical cup? Give a reason for your answer.

(f) For the cup you have chosen in part **(e)**, find the measure of the angle *AOB* that must be cut from the circular disc in order to make the cup.
Give your answer in degrees, correct to the nearest degree.



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Question 9 **(50 marks)**

The *atmospheric pressure* is the pressure exerted by the air in the earth's atmosphere. It can be measured in kilopascals (kPa). The average atmospheric pressure varies with altitude: the higher up you go, the lower the pressure is.

Some students are investigating this variation in pressure, using some data that they found on the internet. They have information about the average pressure at various altitudes.

Six of the entries in the data set are as shown in the table below:

altitude (km)	0	1	2	3	4	5
pressure (kPa)	101·3	89·9	79·5	70·1	61·6	54·0

By looking at the pattern, the students are trying to find a suitable model to match the data.

(a) Hannah suggests that this is approximately a geometric sequence. She says she can match the data fairly well by taking the first term as 101·3 and the common ratio as 0·883.

(i) Complete the table below to show the values given by Hannah's model, correct to one decimal place.

altitude (km)	0	1	2	3	4	5
pressure (kPa)	101·3					

(ii) By considering the percentage errors in the above values, insert an appropriate number to complete the statement below.

"Hannah's model is accurate to within ________%."

(b) Thomas suggests modelling the data with the following exponential function:

$$p = 101{\cdot}3 \times e^{-0{\cdot}1244h}$$

where p is the pressure in kilopascals, and h is the altitude in kilometres.

(i) Taking any **one** value other than 0 for the altitude, verify that the pressure given by Thomas's model and the pressure given by Hannah's model differ by less than 0·01 kPa.

(ii) Explain how Thomas might have arrived at the value of the constant 0·1244 in his model.

(c) Hannah's model is *discrete*, while Thomas's is *continuous*.

(i) Explain what this means.

(ii) State one advantage of a continuous model over a discrete one.

(d) Use Thomas's model to estimate the atmospheric pressure at the altitude of the top of Mount Everest: 8848 metres.

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(e) Using Thomas's model, find an estimate for the altitude at which the atmospheric pressure is half of its value at sea level (altitude 0 km).

(f) People sometimes experience a sensation in their ears when the pressure changes. This can happen when travelling in a fast lift in a tall building. Experiments indicate that many people feel such a sensation if the pressure changes rapidly by 1 kilopascal or more. Suppose that such a person steps into a lift that is close to sea level. Taking a suitable approximation for the distance between two floors, estimate the number of floors that the person would need to travel in order to feel this sensation.

You may use this page for extra work.

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