Coimisiún na Scrúduithe Stáit
State Examinations Commission

Leaving Certificate 2020

Marking Scheme

Applied Mathematics

Higher Level

Note to teachers and students on the use of published marking schemes

Marking schemes published by the State Examinations Commission are not intended to be standalone documents. They are an essential resource for examiners who receive training in the correct interpretation and application of the scheme. This training involves, among other things, marking samples of student work and discussing the marks awarded, so as to clarify the correct application of the scheme. The work of examiners is subsequently monitored by Advising Examiners to ensure consistent and accurate application of the marking scheme. This process is overseen by the Chief Examiner, usually assisted by a Chief Advising Examiner. The Chief Examiner is the final authority regarding whether or not the marking scheme has been correctly applied to any piece of candidate work.

Marking schemes are working documents. While a draft marking scheme is prepared in advance of the examination, the scheme is not finalised until examiners have applied it to candidates' work and the feedback from all examiners has been collated and considered in light of the full range of responses of candidates, the overall level of difficulty of the examination and the need to maintain consistency in standards from year to year. This published document contains the finalised scheme, as it was applied to all candidates' work.

In the case of marking schemes that include model solutions or answers, it should be noted that these are not intended to be exhaustive. Variations and alternatives may also be acceptable. Examiners must consider all answers on their merits, and will have consulted with their Advising Examiners when in doubt.

Future Marking Schemes

Assumptions about future marking schemes on the basis of past schemes should be avoided. While the underlying assessment principles remain the same, the details of the marking of a particular type of question may change in the context of the contribution of that question to the overall examination in a given year. The Chief Examiner in any given year has the responsibility to determine how best to ensure the fair and accurate assessment of candidates' work and to ensure consistency in the standard of the assessment from year to year. Accordingly, aspects of the structure, detail and application of the marking scheme for a particular examination are subject to change from one year to the next without notice.

General Guidelines

1 Penalties of three types are applied to candidates' work as follows:

Slips	- numerical slips	S(-1)
Blunders	- mathematical errors	B(-3)
Misreading	- if not serious	M(-1)

Serious blunder or omission or misreading which oversimplifies:
- award the attempt mark only.

Attempt marks are awarded as follows: 5 (att 2)

2 The marking scheme shows one correct solution to each question.
In many cases there are other equally valid methods.

1. **(a)** A car is travelling on a straight level road at a uniform speed of 26 m s^{-1} when the driver notices a tractor 91·2 m ahead.
The tractor is travelling at a uniform speed of 6 m s^{-1} in the same direction as the car.
The driver of the car hesitates for t seconds before applying the brake.
The maximum deceleration of the car is 5 m s^{-2}.

Find the maximum value of t which would avoid a collision between the car and the tractor.

Car:

$$s = ut + \frac{1}{2}at^2$$
$$s = 26t + 0$$
$$s = 26t \qquad (5)$$

$$v = u + at$$
$$6 = 26 - 5t_1$$
$$t_1 = 4 \qquad (5)$$

$$s = ut + \frac{1}{2}at^2$$
$$s = 26 \times 4 + \frac{1}{2} \times (-5) \times 4^2$$
$$s = 64 \qquad (5)$$

$$s_C = 26t + 64$$

Tractor:

$$s = ut + \frac{1}{2}at^2$$
$$s_T = 6(t + 4) + 0$$
$$s_T = 6t + 24 \qquad (5)$$

$$s_C = s_T + 91.2$$
$$26t + 64 = 6t + 24 + 91.2$$
$$20t = 51.2$$
$$t = 2.56 \qquad (5) \qquad (25)$$

(b) A 60 gram mass is projected vertically upwards with an initial speed of 15 m s^{-1} and half a second later a 40 gram mass is projected vertically upwards from the same point with an initial speed of 22·65 m s^{-1}.

(i) Calculate the height at which the masses will collide.

The masses coalesce on colliding.

(ii) Find the greatest height which the combined mass will reach.

(i)

$$15t - 4.9t^2 = 22.65\left(t - \tfrac{1}{2}\right) - 4.9\left(t - \tfrac{1}{2}\right)^2 \qquad (5)$$

$$12.55t = 12.55$$
$$t = 1 \qquad (5)$$

$$h = 15 \times 1 + \tfrac{1}{2}(-9.8)1^2$$
$$h = 10.1 \text{ m} \qquad (5)$$

(ii)

$$v_1 = 15 - 9.8(1)$$
$$v_1 = 5.2$$

$$v_2 = 22.65 - 9.8\left(1 - \tfrac{1}{2}\right)$$
$$v_2 = 17.75$$

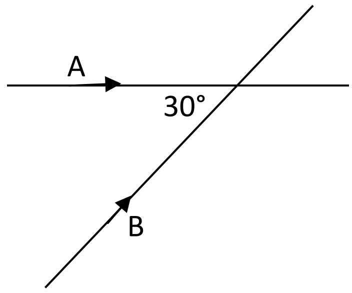
$$0.06 \times 5.2 + 0.04 \times 17.75 = 0.1v$$
$$1.022 = 0.1v$$
$$v = \frac{1.022}{0.1} = \frac{511}{50} = 10.22 \qquad (5)$$

$$v^2 = u^2 + 2as$$
$$0 = 10.22^2 + 2(-9.8)s$$

$$s = 5.3$$

$$\text{greatest height} = 5.3 + 10.1 = 15.4 \text{ m} \qquad (5) \qquad (25)$$

2. **(a)** Two straight roads intersect at an angle of 30°. Car A is moving along one road towards the intersection with a uniform speed of 6 m s^{-1}. Car B is moving along the other road towards the intersection with a uniform speed of 8 m s^{-1}.



(i) Find the velocity of B relative to A.

A reaches the intersection 5 seconds before B.

(ii) Find the distance of each car from the intersection when they are nearest to each other.

(i)

$\overrightarrow{V_A} = 6\ \vec{i}$ (5)

$\overrightarrow{V_B} = 8\cos 30\ \vec{i}\ + 8\sin 30\ \vec{j}$ (5)

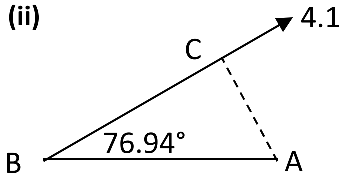
$\overrightarrow{V_{BA}} = \overrightarrow{V_B} - \overrightarrow{V_A}$

$\overrightarrow{V_{BA}} = \left(4\sqrt{3} - 6\right)\ \vec{i}\ + 4\ \vec{j}$ (5)

$\left|\overrightarrow{V_{BA}}\right| = \sqrt{\left(4\sqrt{3} - 6\right)^2 + 4^2} = 4.1$ m s^{-1}

$\tan^{-1}\left(\frac{4}{4\sqrt{3}-6}\right) = 76.94°$ (5)

(ii)



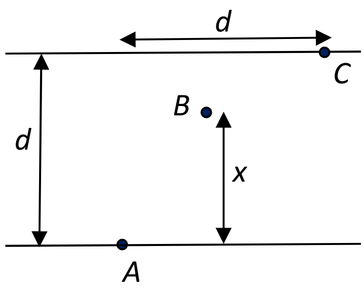
$|BA| = 6 \times 5 = 30$

$t_{BC} = \frac{30\cos 76.94}{4.1} = 1.65\ \text{ s}$ (5)

$s_A = 30 + 6 \times 1.65 = 39.9\ \text{ m}$

$s_B = 8 \times 1.65 = 13.2\ \text{ m}$ (5) (30)

(b) A woman can swim at $\frac{5}{8}$ m s^{-1} in still water.
She swims across a river of width d.
She sets out at right angles to the bank.
The current flows parallel to the straight banks.
It flows at $\frac{1}{2}$ m s^{-1} for the first part of her journey and at 1 m s^{-1} for the remainder of her journey.
She starts from point A, the current changes when she reaches point B and she lands at point C.



She finds that when she has reached the other side, she has drifted downstream a distance equal to the width of the river.
B is a distance x from the bank, as shown in the diagram.

Find x in terms of d.

Vert: $\frac{5}{8}t_1 + \frac{5}{8}t_2 = d$ (5)

$2t_1 + 2t_2 = \frac{16d}{5}$

Horiz: $\frac{1}{2}t_1 + t_2 = d$ (5)

$t_1 + 2t_2 = 2d$

$\Longrightarrow t_1 = \frac{6d}{5}$ (5)

$x = \frac{5}{8}t_1 = \frac{5}{8} \times \frac{6d}{5} = \frac{3}{4}d$ (5) (20)

3. **(a)** A particle is projected from a point P with speed u m s^{-1} at an angle α to the horizontal.

(i) Show that the range of the particle is $\frac{2u^2 \sin\alpha \cos\alpha}{g}$.

The particle is 24·5 m above the horizontal ground after 5 seconds and it strikes the ground 235·2 m from P.

(ii) Find the value of u.

(i)

$$r_j = 0$$

$$u\sin\alpha \times t - \frac{1}{2}gt^2 = 0$$

$$t = \frac{2u\sin\alpha}{g} \qquad (5)$$

$$\text{Range} = u\cos\alpha \times t$$

$$= u\cos\alpha \times \frac{2u\sin\alpha}{g}$$

$$= \frac{2u^2\sin\alpha\cos\alpha}{g} \qquad (5)$$

(ii)

$$r_j = 24.5$$

$$u\sin\alpha \times 5 - 4.9 \times 5^2 = 24.5$$

$$u\sin\alpha = 29.4 \qquad (5)$$

$$\text{Range} = u\cos\alpha \times \frac{2u\sin\alpha}{g} = 235.2$$

$$u\cos\alpha \times \frac{58.8}{g} = 235.2$$

$$u\cos\alpha = 39.2 \qquad (5)$$

$$(u\sin\alpha)^2 + (u\cos\alpha)^2 = 29.4^2 + 39.2^2$$

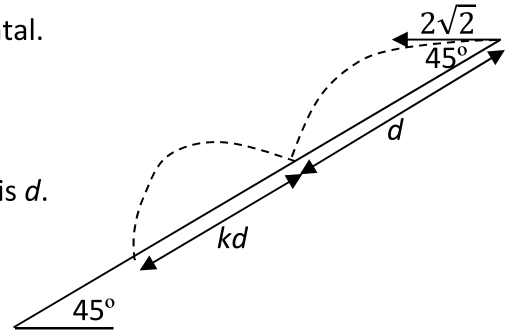
$$u^2(\sin^2\alpha + \cos^2\alpha) = 2401$$

$$u^2 = 2401$$

$$u = 49 \ \text{ m s}^{-1} \qquad (5) \qquad (25)$$

(b) A plane is inclined at an angle of 45° to the horizontal. A particle is projected down the plane with initial speed $2\sqrt{2}$ m s^{-1} at an angle of 45° to the inclined plane.
The range along the inclined plane of the first hop is d.

The plane of projection is vertical and contains the line of greatest slope.



(i) Find the value of d.

The coefficient of restitution between the particle and the inclined plane is 0·4.
The range of the second hop is kd.

(ii) Find the value of k.

(i)

$$r_j = 0 \qquad (5)$$

$$2\sqrt{2}\sin 45 \times t - \frac{1}{2} g \cos 45 \times t^2 = 0$$

$$t = \frac{4\sqrt{2}}{g} = 0.58 \qquad (5)$$

$$d = 2\sqrt{2}\cos 45 \times t + \frac{1}{2} g \sin 45 \times t^2$$
$$= 2\sqrt{2}\cos 45 \times \frac{4\sqrt{2}}{g} + \frac{1}{2} g \sin 45 \times \left(\frac{4\sqrt{2}}{g}\right)^2$$
$$= 2\sqrt{2}\cos 45 \times \frac{4\sqrt{2}}{g} + \frac{1}{2} g \sin 45 \times \left(\frac{4\sqrt{2}}{g}\right)^2$$
$$= \frac{16\sqrt{2}}{g} = \frac{80\sqrt{2}}{49} = 2.31 \qquad (5)$$

(ii)

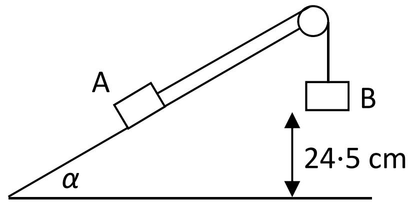
$$v_i = 2\sqrt{2}\cos 45 + g \sin 45 \times \frac{4\sqrt{2}}{g} = 6$$
$$v_j = 2\sqrt{2}\sin 45 - g \cos 45 \times \frac{4\sqrt{2}}{g} = -2$$
$$v_R = 6\ \vec{i} + 0.8\ \vec{j} \qquad (5)$$

$$r_j = 0$$
$$0.8 \times t - \frac{1}{2} g \cos 45 \times t^2 = 0$$
$$t = \frac{1.6\sqrt{2}}{g}$$

$$kd = 6 \times t + \frac{1}{2} g \sin 45 \times t^2$$
$$k \times \frac{16\sqrt{2}}{g} = 6 \times \frac{1.6\sqrt{2}}{g} + \frac{1}{2} g \sin 45 \times \left(\frac{1.6\sqrt{2}}{g}\right)^2$$
$$10k = 6 + 0.8$$

$$k = 0.68 \qquad (5) \qquad (25)$$

4. **(a)** A block A of mass 10*m* on a smooth plane inclined at an angle α with the horizontal, where $\tan\alpha = \frac{3}{4}$, is connected by a light inextensible string which passes over a smooth pulley to a second block B of mass 10*m*. B is 24·5 cm above an inelastic horizontal floor, as shown in the diagram.



The system is released from rest.

Find

(i) the acceleration of B

(ii) the time that B remains in contact with the floor.

(i)

$10mg - T = 10ma$ (5)

$T - 10mg \times \frac{3}{5} = 10ma$ (5)

$a = \frac{g}{5} = 1.96$ (5)

(ii) B

$v^2 = u^2 + 2as$

$v^2 = 0 + 2(1.96)(0.245)$

$v = 0.98$ (5)

A

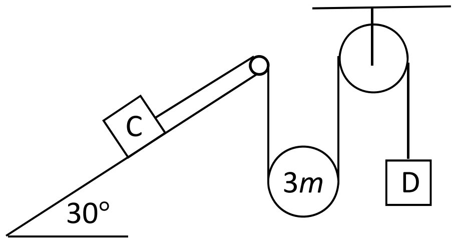
$s = ut + \frac{1}{2}at^2$

$0 = 0.98t + \frac{1}{2}(-g\sin\alpha)t^2$

$0 = 0.98 + \frac{1}{2}(-g \times 0.6)t$

$t = \frac{1}{3} = 0.33$ s. (5) (25)

(b) A particle C of mass $2m$ rests on a rough plane which is inclined at 30° to the horizontal. The coefficient of friction between C and the plane is $\frac{\sqrt{3}}{21}$. A light inextensible string which passes under a smooth movable pulley of mass $3m$ connects C to a particle D of mass m, as shown in the diagram.

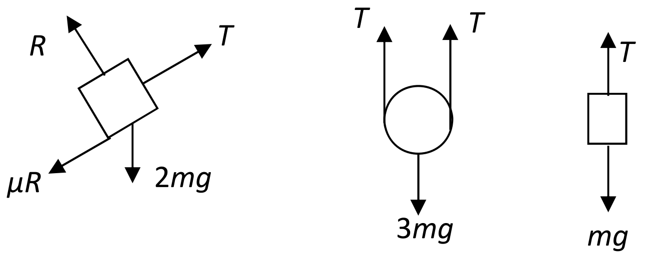


The system is released from rest. C moves up the plane.

(i) Show, on separate diagrams, the forces acting on the moveable pulley and on each of the masses.

(ii) Find in terms of m the tension in the string.

(i)



(5)

(ii)

$$T - 2mg\sin 30 - \mu \times 2mg\cos 30 = 2mp \qquad (5)$$

$$T - mg = mq \qquad (5)$$

$$3mg - 2T = 3m \times \tfrac{1}{2}(p + q) \qquad (5)$$

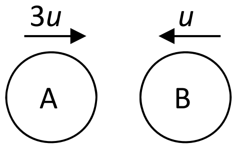
$$3mg - 2T = \tfrac{3}{2}m\left\{\left(\tfrac{T}{2m} - \tfrac{g}{2} - \tfrac{g}{14}\right) + \left(\tfrac{T}{m} - g\right)\right\}$$

$$3mg - 2T = \tfrac{3}{4}T - \tfrac{6}{7}mg + \tfrac{3}{2}T - \tfrac{3}{2}mg$$

$$3mg + \tfrac{6}{7}mg + \tfrac{3}{2}mg = 2T + \tfrac{3}{4}T + \tfrac{3}{2}T$$

$$T = \tfrac{150}{119}mg = \tfrac{210}{17}m \qquad (5) \qquad (25)$$

5. **(a)** A smooth sphere A of mass *m*, moving with speed 3*u* on a smooth horizontal table collides directly with a smooth sphere B of mass 2*m*, moving in the opposite direction with speed *u*. The directions of motion of A and B are reversed by the collision.

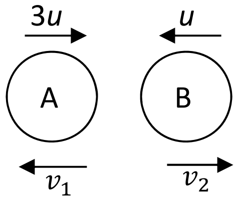


The coefficient of restitution between A and B is *e*.

(i) Find the speed, in terms of *u* and *e*, of each sphere after the collision.

Subsequently B hits a wall at right angles to the line of motion of A and B.
The coefficient of restitution between B and the wall is $\frac{1}{2}$.
After B rebounds from the wall there is a further collision between A and B.

(ii) Show that $\frac{1}{8} < e < \frac{1}{4}$.



(i) PCM $m(3u) + 2m(-u) = m(-v_1) + 2mv_2$ (5)

NEL $-v_1 - v_2 = -e(3u + u)$ (5)

$-v_1 + 2v_2 = u$

$v_1 + v_2 = 4eu$

$v_1 = \frac{u(8e-1)}{3}$ $v_2 = \frac{u(1+4e)}{3}$ (5), (5)

(ii) $\frac{1}{2}v_2 = \frac{u(1+4e)}{6}$

2[nd] collision if $\frac{1}{2}v_2 > v_1$

$\frac{u(1+4e)}{6} > \frac{u(8e-1)}{3}$

$1 + 4e > 16e - 2$

$e < \frac{1}{4}$ (5)

$v_1 > 0$

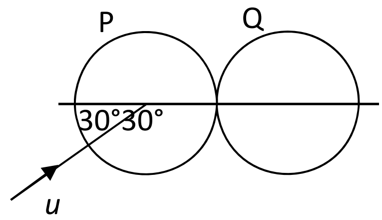
$\frac{u(8e-1)}{3} > 0$

$8e - 1 > 0$

$e > \frac{1}{8}$

$\frac{1}{8} < e < \frac{1}{4}$ (5) (30)

(b) A smooth sphere P has mass m_1 and speed u.
It collides obliquely with a smooth sphere Q, of mass m_2, which is at rest.
Before the collision the direction of P makes an angle of 30° to the line of centres, as shown in the diagram.



The coefficient of restitution between the spheres is e.

Prove that P will turn through a right-angle if $4m_1 = (3e-1)m_2$.

P	m_1	$\frac{u\sqrt{3}}{2}\ \vec{i} + \frac{u}{2}\ \vec{j}$	$v_1\ \vec{i} + \frac{u}{2}\ \vec{j}$
Q	m_2	$0\ \vec{i} + 0\ \vec{j}$	$v_2\ \vec{i} + 0\ \vec{j}$

PCM $\quad m_1\left(\frac{u\sqrt{3}}{2}\right) + m_2(0) = m_1v_1 + m_2v_2$ (5)

NEL $\quad v_1 - v_2 = -e\left(\frac{u\sqrt{3}}{2} - 0\right)$ (5)

$$\frac{m_1}{m_2}v_1 + v_2 = \frac{m_1}{m_2}\left(\frac{u\sqrt{3}}{2}\right)$$

$$v_1 - v_2 = -e\left(\frac{u\sqrt{3}}{2}\right)$$

$$\frac{m_1}{m_2}v_1 + v_1 = \frac{m_1}{m_2}\left(\frac{u\sqrt{3}}{2}\right) - e\left(\frac{u\sqrt{3}}{2}\right) \quad (5)$$

$$\tan 60 = \frac{\frac{u}{2}}{-v_1}$$

$$v_1 = -\frac{u}{2\sqrt{3}} = -\frac{u\sqrt{3}}{6}$$

$$\frac{m_1}{m_2}\left(-\frac{u}{2\sqrt{3}}\right) - \frac{u}{2\sqrt{3}} = \frac{m_1}{m_2}\left(\frac{u\sqrt{3}}{2}\right) - e\left(\frac{u\sqrt{3}}{2}\right)$$

$$\frac{m_1}{m_2} + 1 = -\frac{m_1}{m_2}(3) + \mathrm{e}(3)$$

$$4m_1 = (3e-1)m_2 \quad (5) \quad (20)$$

6. **(a)** A particle D of mass *m* is suspended from a fixed point by a light elastic string of natural length ℓ and elastic constant $\frac{6mg}{\ell}$.
Initially D rests in equilibrium with the string vertical.
The particle is now pulled down a vertical distance $\frac{1}{3}\ell$ below its equilibrium position and released from rest.

(i) Show that D moves with simple harmonic motion.

(ii) In terms of ℓ, find the greatest speed of D while the string remains taut.

(i)

$T_0 = mg$ (5)

$ke = mg$

$\frac{6mg}{\ell}e = mg$

$e = \frac{1}{6}\ell$ (5)

$m\ddot{x} = mg - T$

$m\ddot{x} = mg - \frac{6mg}{\ell}(e + x)$

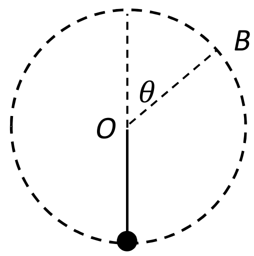
$\ddot{x} = -\frac{6g}{\ell}x \quad \Longrightarrow \quad \text{S.H.M.}$ (5)

(ii)

$\omega = \sqrt{\frac{6g}{\ell}}$ (5)

$\text{greatest speed} = \omega \times \frac{\ell}{3} = \frac{1}{3}\sqrt{6\ell g}$ (5) (25)

(b) A particle P is attached to one end of a light inextensible string of length d.
The other end of the string is attached to a fixed point *O*.
The particle is hanging freely at rest, with the string vertical, when it is projected horizontally with speed $\sqrt{3gd}$.
The particle moves in a vertical circle.
The string becomes slack when P is at the point *B*.
OB makes an angle θ with the upward vertical.



(i) Show that $\cos\theta = \frac{1}{3}$.

(ii) In terms of d, find the greatest height of P above *B* in the subsequent motion.

(i)

$$T + mg\cos\theta = \frac{mv^2}{d} \qquad (5)$$

$$\frac{1}{2}m(3gd) = \frac{1}{2}mv^2 + mg(d + d\cos\theta) \qquad (5)$$

$$mg(1 - 2\cos\theta) = \frac{mv^2}{d}$$

$$T + mg\cos\theta = \frac{mv^2}{d} = mg(1 - 2\cos\theta)$$

$$T + 3mg\cos\theta = mg$$

$$T = 0 \quad \Rightarrow \quad \cos\theta = \frac{1}{3} \qquad (5)$$

(ii)

$$T + mg\cos\theta = \frac{mv^2}{d}$$

$$0 + mg \times \frac{1}{3} = \frac{mv^2}{d}$$

$$v^2 = \frac{1}{3}dg \qquad (5)$$

$$v^2 = u^2 + 2as$$

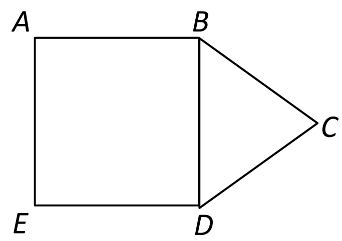
$$0 = (v\sin\theta)^2 - 2gs$$

$$0 = \frac{1}{3}dg \times \frac{8}{9} - 2gs$$

$$s = \frac{4d}{27} \qquad (5) \qquad (25)$$

7. **(a)** A uniform lamina *ABCDE* is formed by joining square *ABDE* with triangle *BCD*, as shown in the diagram.
BCD is an isosceles triangle.

$|AB| = 24$ cm and $|BC| = 20$ cm.



(i) Find the distance of the centre of gravity of the lamina from *AE*.

The lamina is freely suspended from *B* and hangs in equilibrium.

(ii) Find the angle which *BD* makes with the vertical.

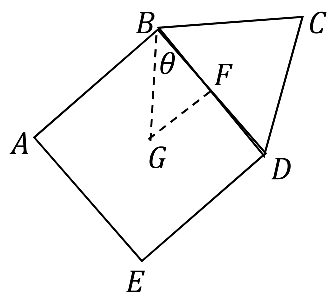
(i)

	Area	c.g. from *AE*	
EDBA	$24 \times 24 = 576$	12	(5)
DCB	$\frac{1}{2} \times 24 \times 16 = 192$	$24 + \frac{1}{3} \times 16 = \frac{88}{3}$	(5)
EDCBA	$576 + 192 = 768$	x	(5)

$$768x = 576 \times 12 + 192 \times \frac{88}{3}$$

$$x = \frac{49}{3} = 16.3 \text{ cm}$$ (5)

(ii)



$$|GF| = 24 - \frac{49}{3} = \frac{23}{3}$$

$$|BF| = 12$$

$$\tan\theta = \frac{23}{3} \div 12 = \frac{23}{36}$$

$$\theta = 32.57°.$$ (5) (25)

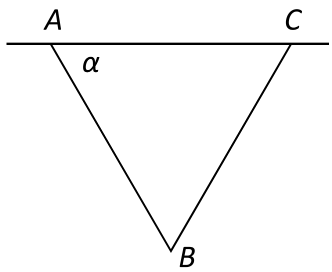
(b) Two uniform rods *AB* and *BC*, of equal length, and of weights 2*W* and *W* respectively are smoothly hinged at *B*.

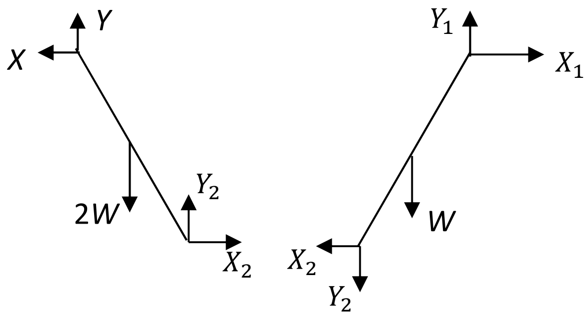
A and *C* are hinged to a horizontal beam.

The rods are in a vertical plane with *B* below *AC*.

$|\angle BAC| = \alpha$.

Find the horizontal and vertical components of the reaction of the hinge at *B* on the rod *AB*.





AB $\circlearrowright A$

$$X_2 \times l\sin\alpha + Y_2 \times l\cos\alpha = 2W \times \tfrac{1}{2} l\cos\alpha \qquad (5), (5)$$

$$X_2 \tan\alpha + Y_2 = W$$

BC $\circlearrowright C$

$$X_2 \times l\sin\alpha = Y_2 \times l\cos\alpha + W \times \tfrac{1}{2} l\cos\alpha \qquad (5)$$

$$X_2 \tan\alpha = Y_2 + \tfrac{1}{2}W$$

$$Y_2 = W - Y_2 - \tfrac{1}{2}W$$

$$Y_2 = \tfrac{1}{4}W \qquad (5)$$

$$X_2 \tan\alpha + \tfrac{1}{4}W = W$$

$$X_2 \tan\alpha = \tfrac{3}{4}W$$

$$X_2 = \frac{3W}{4\tan\alpha} \qquad (5) \qquad (25)$$

8. **(a)** Prove that the moment of inertia of a uniform disc, of mass *m* and radius *r* about an axis through its centre, perpendicular to its plane, is $\frac{1}{2}mr^2$.

Let M = mass per unit area

mass of element = $M\{2\pi x dx\}$

moment of inertia of the element = $M\{2\pi x dx\}\, x^2$ (5)

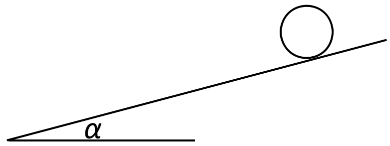
moment of inertia of the disc = $2\pi M \int_0^r x^3\, dx$ (5)

= $2\pi M \left[\frac{x^4}{4}\right]_0^r$ (5)

= $\frac{1}{2}\pi M r^4$

= $\frac{1}{2}mr^2$ (5) (20)

(b) A wheel consists of a uniform circular disc of radius *r*.
The wheel rolls down a rough inclined plane without slipping.
The plane is inclined at an angle α to the horizontal.



(i) Show that the acceleration of the wheel is $\frac{2}{3}g\sin\alpha$.

The coefficient of friction between the wheel and the inclined plane is 0·2.

(ii) Find the maximum value of α.

(i)

$$\frac{1}{2}I\omega^2 + \frac{1}{2}mv^2 = mgh \qquad (5)$$

$$\frac{1}{2}\left(\frac{1}{2}mr^2\right)\left(\frac{v}{r}\right)^2 + \frac{1}{2}mv^2 = mgh \qquad (5)$$

$$\frac{3}{4}mv^2 = mgh$$

$$v^2 = \frac{4}{3}gh = \frac{4}{3}gx\sin\alpha \qquad (5)$$

$$v^2 = u^2 + 2as$$

$$\frac{4}{3}gx\sin\alpha = 0 + 2ax$$

$$a = \frac{2}{3}g\sin\alpha \qquad (5)$$

(ii)

$$ma = mg\sin\alpha - \mu mg\cos\alpha \qquad (5)$$

$$m \times \frac{2}{3}g\sin\alpha = mg\sin\alpha - 0.2mg\cos\alpha$$

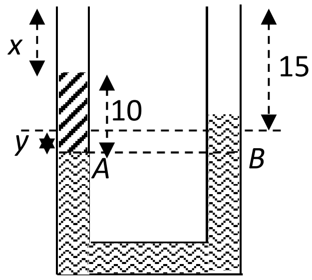
$$\frac{1}{3}\sin\alpha = 0.2\cos\alpha$$

$$\tan\alpha = 0.6$$

$$\alpha = 30.96°. \qquad (5) \qquad (30)$$

9. **(a)** A U-tube whose limbs are vertical and of equal length contains mercury of relative density 13·6.
The surface of the mercury is 15 cm from the top of each limb.
The cross-sectional area of the U-tube is 10 cm^2.
100 cm^3 of oil, of relative density 0·68, is poured into one limb. The surface of the oil is *x* cm from the top of the limb.

Find the value of *x*.



$$h = \frac{V}{A} = \frac{100}{10} = 10 \text{ cm} \qquad (5)$$

$$P_A = 680 \times g \times 10 \times 10^{-2} \qquad (5)$$

$$P_B = 13600 \times g \times 2y \times 10^{-2} \qquad (5)$$

$$P_A = P_B$$

$$680 \times g \times 10 \times 10^{-2} = 13600 \times g \times 2y \times 10^{-2}$$

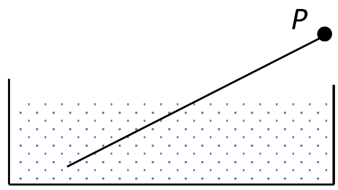
$$y = 0.25 \text{ cm} \qquad (5)$$

$$x + 10 - 15 = y$$

$$x - 5 = 0.25$$

$$x = 5.25 \text{ cm.} \qquad (5) \qquad (25)$$

(b) A uniform rod, of length ℓ and weight W, is freely hinged at the point P.
The rod is free to move about a horizontal axis through P.
The other end of the rod is immersed in water.
The relative density of the rod is 0·64.
The rod is in equilibrium and is inclined as shown in the diagram.

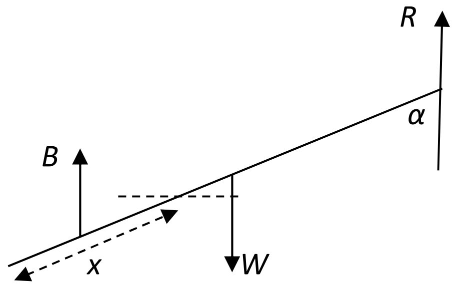


Find

(i) the length of the immersed part of the rod in terms of ℓ

(ii) the reaction at the hinge in terms of W.

[Density of water = 1000 kg m^{-3}]



(i) $B = \dfrac{\frac{x}{\ell}W}{0.64} = \dfrac{xW}{0.64\ell}$ or $B = 1000xAg$ (5)

$B\left\{\ell - \frac{1}{2}x\right\}\sin\alpha = W\frac{1}{2}\ \ell\sin\alpha$ (5)

$\dfrac{xW}{0.64\ell}\left\{\ell - \frac{1}{2}x\right\}\sin\alpha = W\frac{1}{2}\ \ell\sin\alpha$

$2x\left\{\ell - \frac{1}{2}x\right\} = 0.64\ \ell^2$

$0.64\ \ell^2 - 2x\ell + x^2 = 0$

$(0.4\ell - x)(1.6\ell - x) = 0$

$x = 0.4\ell$ (5)

(ii) $R + B = W$ (5)

$R + \dfrac{xW}{0.64\ell} = W$

$R + \frac{5}{8}W = W$

$R = \frac{3}{8}W$ (5) (25)

10. **(a)** One method of dyeing a piece of cloth is to immerse it in a container which has *P* grams of dye dissolved in a fixed volume of water.

The cloth absorbs the dye at a rate proportional to the mass of dye remaining.

$$\frac{dx}{dt} = k(P - x)$$

where *t* is time in seconds, *x* is the mass of dye absorbed by the cloth and $k = \frac{1}{50}$.

(i) Find the time taken to dye a piece of cloth if a mass of $\frac{5}{8}P$ needs to be absorbed to reach the desired colour.
(Note: $\int \frac{dx}{a+bx} = \frac{1}{b}\ln|a + bx| + c$)

An alternative method is to keep the mass of dye present in the water constant at *P* grams by continuously adding dye throughout the process.

(ii) Find the time taken to dye the piece of cloth to the desired colour using this method.

(i)

$$\frac{dx}{dt} = \frac{P-x}{50}$$

$$\int \frac{dx}{P-x} = \frac{1}{50}\int dt \qquad (5)$$

$$[-\ln(P-x)]_0^{\frac{5}{8}P} = \frac{1}{50}[t]_0^t \qquad (5)$$

$$-\ln\left(\frac{3}{8}P\right) + \ln P = \frac{1}{50}t$$

$$\ln\frac{8}{3} = \frac{1}{50}t$$

$$t = 50\ln\frac{8}{3} = 49.0 \text{ s.} \qquad (5)$$

(ii)

$$\frac{dx}{dt} = \frac{P}{50}$$

$$\int dx = \frac{1}{50}P\int dt$$

$$[x]_0^{\frac{5}{8}P} = \frac{1}{50}P[t]_0^t \qquad (5)$$

$$\frac{5}{8}P = \frac{1}{50}P \times t$$

$$t = \frac{250}{8} = 31.25 \text{ s.} \qquad (5) \qquad (25)$$

(b) A particle P travelling in a straight line has a deceleration of $4v^{n+1}$ m s^{-2}, where n (> 0) is a constant and v is its speed at time t (> 0).

P has an initial speed of u.

(i) Find an expression for v in terms of u, n and t.

(ii) When $n = 3$ obtain an expression for the speed of P when it has travelled a distance of 3 m from its initial position.

(i)

$$\frac{dv}{dt} = -4v^{n+1}$$

$$-\int \frac{dv}{v^{n+1}} = 4\int dt \qquad (5)$$

$$\frac{1}{n} \times \left[\frac{1}{v^n}\right]_u^v = 4[t]_0^t \qquad (5)$$

$$\frac{1}{v^n} - \frac{1}{u^n} = 4nt$$

$$v^n = \frac{1}{4nt + \frac{1}{u^n}}$$

$$v = \left(\frac{1}{4nt + \frac{1}{u^n}}\right)^{\frac{1}{n}} = \left(\frac{u^n}{4ntu^n + 1}\right)^{\frac{1}{n}} = \frac{u}{(4ntu^n + 1)^{\frac{1}{n}}} \qquad (5)$$

(ii)

$$v\frac{dv}{ds} = -4v^4$$

$$-\int \frac{dv}{v^3} = 4\int ds$$

$$\left[\frac{1}{2v^2}\right]_u^v = [4s]_0^3 \qquad (5)$$

$$\frac{1}{2v^2} - \frac{1}{2u^2} = 12 - 0$$

$$\frac{1}{2v^2} = \frac{1}{2u^2} + 12 = \frac{1 + 24u^2}{2u^2}$$

$$v^2 = \frac{u^2}{1 + 24u^2}$$

$$v = \frac{u}{\sqrt{1 + 24u^2}} \qquad (5) \qquad (25)$$