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# Coimisiún na Scrúduithe Stáit
# State Examinations Commission

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## LEAVING CERTIFICATE EXAMINATION, 2006

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## MATHEMATICS – HIGHER LEVEL

**PAPER 2 ( 300 marks )**

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**MONDAY, 12 JUNE – MORNING, 9:30 to 12:00**

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Attempt **FIVE** questions from **Section A** and **ONE** question from **Section B**.
Each question carries 50 marks.

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WARNING: **Marks will be lost if all necessary work is not clearly shown.**

**Answers should include the appropriate units of measurement, where relevant.**

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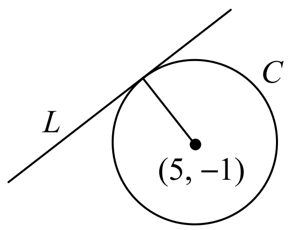
## SECTION A

**Answer FIVE questions from this section.**

**1.** **(a)** $a\ (-1, -3)$ and $b\ (3, 1)$ are the end-points of a diameter of a circle. Write down the equation of the circle.

**(b)** Circle $C$ has centre $(5, -1)$. The line $L$: $3x - 4y + 11 = 0$ is a tangent to $C$.




**(i)** Show that the radius of $C$ is 6.

**(ii)** The line $x + py + 1 = 0$ is also a tangent to $C$. Find two possible values of $p$.

**(c)** $S$ is the circle $x^2 + y^2 + 4x + 4y - 17 = 0$ and $K$ is the line $4x + 3y = 12$.

**(i)** Show that the line $K$ does not intersect $S$.

**(ii)** Find the co-ordinates of the point on $S$ that is closest to $K$.

**2.** **(a)** $\vec{x} = -3\vec{i} + \vec{j}$. Express $\left(\vec{x}^{\perp}\right)^{\perp}$ in terms of $\vec{i}$ and $\vec{j}$.

**(b)** $\vec{p} = -5\vec{i} + 2\vec{j}$, $\vec{q} = \vec{i} - 6\vec{j}$ and $\vec{r} = -\vec{i} + 5\vec{j}$.

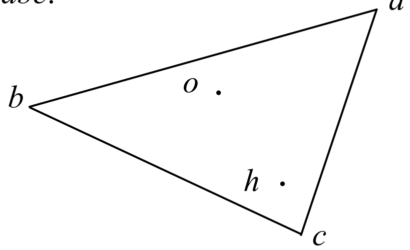
**(i)** Express $\overrightarrow{pq}$ and $\overrightarrow{pr}$ in terms of $\vec{i}$ and $\vec{j}$.

**(ii)** Given that $10\vec{s} = \left|\overrightarrow{pr}\right|\overrightarrow{pq} + \left|\overrightarrow{pq}\right|\overrightarrow{pr}$, express $\vec{s}$ in terms of $\vec{i}$ and $\vec{j}$.

**(iii)** Find the measure of the angle between $\vec{s}$ and $\overrightarrow{pr}$.

**(c)** The origin $o$ is the circumcentre of the triangle $abc$.

If $\vec{h} = \vec{a} + \vec{b} + \vec{c}$, show that $\overrightarrow{ah} \perp \overrightarrow{bc}$.



**3.** **(a)** Show that the line containing the points (3, −6) and (−7, 12) is perpendicular to the line $5x - 9y + 6 = 0$.

**(b)** The line $K$ has positive slope and passes through the point $p$ (2, −9). $K$ intersects the $x$-axis at $q$ and the $y$-axis at $r$ and $|pq| : |pr| = 3:1$. Find the co-ordinates of $q$ and the co-ordinates of $r$.

**(c)** **(i)** Prove that the measure of one of the angles between two lines with slopes $m_1$ and $m_2$ is given by

$$\tan\theta = \frac{m_1 - m_2}{1 + m_1 m_2}.$$

**(ii)** $L$ is the line $y = 4x$ and $K$ is the line $x = 4y$. $f$ is the transformation $(x, y) \to (x', y')$, where $x' = 2x - y$ and $y' = x + 3y$. Find the measure of the acute angle between $f(L)$ and $f(K)$, correct to the nearest degree.

**4.** **(a)** Write down the values of $A$ for which

$$\cos A = \frac{1}{2}, \text{ where } 0^\circ \leq A \leq 360^\circ.$$

**(b)** **(i)** Express $\sin(3x + 60^\circ) - \sin x$ as a product of sine and cosine.

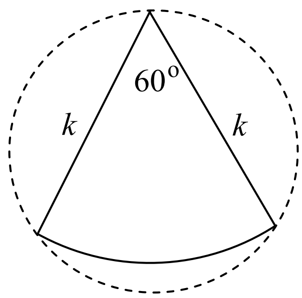
**(ii)** Find all the solutions of the equation

$$\sin(3x + 60^\circ) - \sin x = 0, \text{ where } 0^\circ \leq x \leq 360^\circ.$$

**(c)** The diagram shows a sector (solid line) circumscribed by a circle (dashed line).

**(i)** Find the radius of the circle in terms of $k$.

**(ii)** Show that the circle encloses an area which is double that of the sector.



**5.** **(a)** **(i)** Copy and complete the table below for $f : x \to \tan^{-1} x$, giving the values for $f(x)$ in terms of $\pi$.

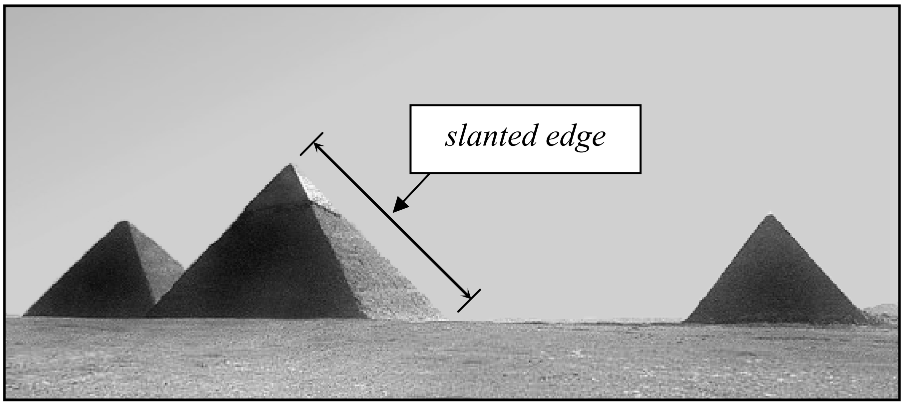
| $x$ | $-\sqrt{3}$ | $-1$ | $-\frac{1}{\sqrt{3}}$ | $0$ | $\frac{1}{\sqrt{3}}$ | $1$ | $\sqrt{3}$ |
|---|---|---|---|---|---|---|---|
| $f(x)$ | | | | | | $\frac{\pi}{4}$ | |

**(ii)** Draw the graph of $y = f(x)$ in the domain $-2 \le x \le 2$, scaling the $y$-axis in terms of $\pi$.

**(iii)** Draw the two horizontal asymptotes of the graph.

**(iv)** For some values of $k \in \mathbf{R}$, but not all values, $\tan^{-1}(\tan k) = k$.
State the range of values of $k$ for which $\tan^{-1}(\tan k) = k$.
Show, by means of an example, what happens outside the range.

**(b)** The great pyramid at Giza in Egypt has a square base and four triangular faces. The base of the pyramid is of side 230 metres and the pyramid is 146 metres high. The top of the pyramid is directly above the centre of the base.




**(i)** Calculate the length of one of the slanted edges, correct to the nearest metre.

**(ii)** Calculate, correct to two significant figures, the total area of the four triangular faces of the pyramid (assuming they are smooth flat surfaces).

**6.** **(a)** **(i)** How many different teams of three people can be chosen from a panel of six boys and five girls?

**(ii)** If the team is chosen at random, find the probability that it consists of girls only?

**(b)** **(i)** Solve the difference equation $6u_{n+2} - 7u_{n+1} + u_n = 0$, where $n \geq 0$, given that $u_0 = 8$ and $u_1 = 3$.

**(ii)** Verify that the solution to part **(i)** also satisfies the difference equation

$$6u_{n+1} - u_n - 10 = 0.$$

**(c)** There are thirty days in June. Seven students have their birthdays in June. The birthdays are independent of each other and all dates are equally likely.

**(i)** What is the probability that all seven students have the same birthday?

**(ii)** What is the probability that all seven students have different birthdays?

**(iii)** Show that the probability that at least two have the same birthday is greater than 0·5.

**7.** **(a)** The password for a mobile phone consists of five digits.

**(i)** How many passwords are possible?

**(ii)** How many of these passwords start with a 2 and finish with an odd digit?

**(b)** For a lottery, 35 cards numbered 1 to 35 are placed in a drum. Five cards will be chosen at random from the drum as the winning combination.

**(i)** How many different combinations are possible?

**(ii)** How many of all the possible combinations will match exactly four numbers with the winning combination?

**(iii)** How many of all the possible combinations will match exactly three numbers with the winning combination?

**(iv)** Show that the probability of matching at least three numbers with the winning combination is approximately 0·014.

**(c)** The mean of the integers from $-n$ to $n$, inclusive, is 0.

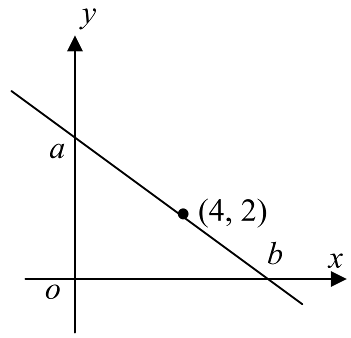
Show that the standard deviation is $\sqrt{\dfrac{n(n+1)}{3}}$.

## SECTION B

**Answer ONE question from this section.**

**8.** **(a)** Derive the Maclaurin series for $f(x) = e^x$ up to and including the term containing $x^3$.

**(b)** A line passes through the point (4, 2) and has slope $m$, where $m < 0$. The line intersects the axes at the points $a$ and $b$.




**(i)** Find the co-ordinates of $a$ and $b$, in terms of $m$.

**(ii)** Hence, find the value of $m$ for which the area of triangle $aob$ is a minimum.

**(c)** Use the ratio test to test each of the following series for convergence. In each case, specify clearly the range of values of $x$ for which the series converges, the range of values for which it diverges, and the value(s) of $x$ for which the test is inconclusive.

**(i)** $\sum_{n=1}^{\infty} n3^n x^n$ **(ii)** $\sum_{n=1}^{\infty} \frac{(n+1)!\,n!}{(2n)!} x^n$.

**9.** **(a)** $z$ is a random variable with standard normal distribution. Find the value of $z_1$ for which $P(z > z_1) = 0{\cdot}0808$.

**(b)** A bag contains the following cardboard shapes:
10 red squares, 15 green squares, 8 red triangles and 12 green triangles.
One of the shapes is drawn at random from the bag.
$E$ is the event that a square is drawn.
$F$ is the event that a green shape is drawn.

**(i)** Find $P(E \cap F)$.

**(ii)** Find $P(E \cup F)$.

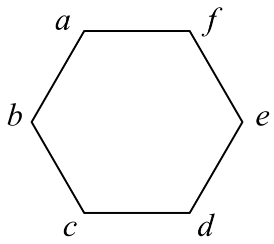
**(iii)** State whether $E$ and $F$ are independent events, giving a reason for your answer.

**(iv)** State whether $E$ and $F$ are mutually exclusive events, giving a reason for your answer.

**(c)** The marks awarded in an examination are normally distributed with a mean mark of 60 and a standard deviation of 10.
A sample of 50 students has a mean mark of 63.
Test, at the 5% level of significance, the hypothesis that this is a random sample from the population.

**10.** **(a)** $G$ is the set of rotations that map a regular hexagon onto itself. $(G, \circ)$ is a group, where $\circ$ denotes composition.

The anti-clockwise rotation through $60^\circ$ is written as $R_{60^\circ}$.




**(i)** List the elements of $G$.

**(ii)** State which elements of the group, if any, are generators.

**(iii)** List all the proper subgroups of $(G, \circ)$.

**(iv)** Find $Z(G)$, the centre of $(G, \circ)$. Justify your answer.

**(b)** **(i)** Show that the group $\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \right\}$ under matrix multiplication is isomorphic to the group $\{0, 1\}$ under addition modulo 2.

**(ii)** Prove that any infinite cyclic group is isomorphic to $(\mathbf{Z}, +)$.

**11.** **(a)** **(i)** Find the image of $a$ $(-1, 2)$ and $b$ $(0, 4)$ under the transformation

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 2 \\ -4 \end{pmatrix}.$$

**(ii)** Show that $ab$ is parallel to $a'b'$.

**(b)** $p$ $(x, y)$ is a point such that the distance from $p$ to the point $(2, 0)$ is half the distance from $p$ to the line $x = 8$.

**(i)** Find the equation of the locus of $p$.

**(ii)** Show that this locus is an ellipse centred at the origin, by expressing its equation in the form $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$.

**(c)** Prove that the areas of all parallelograms circumscribed about a given ellipse at the endpoints of conjugate diameters are equal.

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