

AN ROINN OIDEACHAIS AGUS EOLAÍOCHTA

LEAVING CERTIFICATE EXAMINATION, 2001

MATHEMATICS — HIGHER LEVEL

PAPER 1 (300 marks)

THURSDAY, 7 JUNE — MORNING, 9.30 to 12.00

Attempt **SIX QUESTIONS** (50 marks each).

WARNING: Marks may be lost if all necessary work is not clearly shown.

1. (a) Find the real numbers a and b such that

$$x^2 + 4x - 6 = (x + a)^2 + b \quad \text{for all } x \in \mathbf{R}.$$

- (b) Let $f(x) = 2x^3 + mx^2 + nx + 2$ where m and n are constants.

Given that $x - 1$ and $x + 2$ are factors of $f(x)$, find the value of m and the value of n .

- (c) $x^2 - px + q$ is a factor of $x^3 + 3px^2 + 3qx + r$.

(i) Show that $q = -2p^2$.

(ii) Show that $r = -8p^3$.

(iii) Find the three roots of $x^3 + 3px^2 + 3qx + r = 0$ in terms of p .

2. (a) Solve the simultaneous equations

$$\begin{aligned}x - y &= 0 \\(x + 2)^2 + y^2 &= 10.\end{aligned}$$

- (b) (i) Solve for x

$$|3x + 5| < 4.$$

- (ii) Simplify $\left(x^2 + \sqrt{2} + \frac{1}{x^2}\right)\left(x^2 - \sqrt{2} + \frac{1}{x^2}\right)$ and express your answer in the form $x^n + \frac{1}{x^n}$ where n is a whole number.

- (c) α and β are real numbers such that $\alpha + \beta = -7$ and $\alpha\beta = 11$.

(i) Show that $\alpha^2 + \beta^2 = 27$.

- (ii) Find a quadratic equation with roots $\frac{\alpha}{\beta}$ and $\frac{\beta}{\alpha}$ and write your answer in the form $px^2 + qx + r = 0$ where $p, q, r \in \mathbf{Z}$.

3. (a) Let $u = \frac{1+3i}{3+i}$ where $i^2 = -1$.

(i) Express u in the form $a + ib$ where $a, b \in \mathbf{R}$.

(ii) Evaluate $|u|$.

(b) (i) Write the simultaneous equations

$$x - \sqrt{3}y = -2$$

$$\sqrt{3}x + y = 2\sqrt{3}$$

in the form $A \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -2 \\ 2\sqrt{3} \end{pmatrix}$ where A is a 2×2 matrix.

(ii) Then, find A^{-1} and use it to solve the equations for x and y .

(c) (i) Write $\begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} -2 & 3 \\ -4 & 5 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$ in the form $ax^2 + bxy + cy^2$ where $a, b, c \in \mathbf{Z}$.

(ii) Show that $z^2 - 16$ is a factor of $z^3 + (1+i)z^2 - 16z - 16(1+i)$ and hence, find the three roots of $z^3 + (1+i)z^2 - 16z - 16(1+i) = 0$.

4. (a) The sum of the first n terms of an arithmetic series is given by $S_n = 3n^2 - 4n$.

Use S_n to find (i) the first term, T_1

(ii) the sum of the second term and the third term, $T_2 + T_3$.

(b) (i) Show that $\frac{1}{(n+2)(n+3)} = \frac{1}{n+2} - \frac{1}{n+3}$ for $n \in \mathbf{N}$.

(ii) Hence, find $\sum_{n=1}^k \frac{1}{(n+2)(n+3)}$ and evaluate $\sum_{n=1}^{\infty} \frac{1}{(n+2)(n+3)}$.

(c) (i) Write $\frac{n^3 + 8}{n+2}$ in the form $an^2 + bn + c$ where $a, b, c \in \mathbf{R}$.

(ii) Hence, evaluate $\sum_{n=1}^{30} \frac{n^3 + 8}{n+2}$.

[Note: $\sum_{n=1}^k n = \frac{k}{2}(k+1)$; $\sum_{n=1}^k n^2 = \frac{k}{6}(k+1)(2k+1)$.]

5. (a) The second term, T_2 , of a geometric sequence is 21.
The third term, T_3 , is -63 .

Find (i) the common ratio

(ii) the first term.

- (b) (i) Solve $\log_6(x+5) = 2 - \log_6 x$ for $x > 0$.

(ii) In the binomial expansion of $(1+kx)^6$, the coefficient of x^4 is 240.
Find the two possible real values of k .

- (c) Use induction to prove that, for n a positive integer,

$$(\cos \theta + i \sin \theta)^n = \cos n\theta + i \sin n\theta$$

for all $\theta \in \mathbf{R}$ and $i^2 = -1$.

6. (a) Differentiate $\frac{x}{1+x^2}$ with respect to x .

- (b) (i) Given that $y = \sqrt{x}$, what is $\frac{dy}{dx}$?

(ii) Now, find from first principles the derivative of \sqrt{x} with respect to x .

- (c) Let $x = t^2 e^t$ and $y = t + 2 \ln t$ for $t > 0$.

(i) Find $\frac{dx}{dt}$ and $\frac{dy}{dt}$ in terms of t .

(ii) Hence, show that $\frac{dy}{dx} = \frac{1}{x}$.

7. (a) Taking $x_1 = 1$ as the first approximation to the real root of the equation

$$x^3 + x^2 - 1 = 0,$$

use the Newton-Raphson method to find x_2 , the second approximation.

- (b) (i) Differentiate $\tan^{-1} 7x$ with respect to x .

- (ii) Given that $y = \sin x \cos x$, find $\frac{dy}{dx}$ and express it in the form $\cos nx$ where $n \in \mathbf{Z}$.

- (c) Let $g(x) = x^2 + \frac{a}{x^2}$ where a is a real number and $x \in \mathbf{R}$, $x \neq 0$.

Given that $g(x)$ has a turning point at $x = 2$,

- (i) find the value of a
(ii) prove that $g(x)$ has no local maximum points.

8. (a) Find (i) $\int \frac{1}{x^3} dx$ (ii) $\int \sin 5x dx$.

- (b) Evaluate (i) $\int_0^3 \frac{12}{x^2 + 9} dx$ (ii) $\int_0^4 \frac{(x+4)}{\sqrt{x^2 + 8x + 1}} dx$.

- (c) a is a real number such that $0 < a < 8$.

The line $y = ax$ intersects the curve $y = x(8-x)$ at $x = 0$ and at $x = p$.

- (i) Show that $p = 8 - a$.
(ii) Show that the area between the curve and the line is $\frac{p^3}{6}$ square units.

