Coimisiún na Scrúduithe Stáit

State Examinations Commission

Leaving Certificate 2022

Marking Scheme

Applied Mathematics

Higher Level

Note to teachers and students on the use of published marking schemes

Marking schemes published by the State Examinations Commission are not intended to be standalone documents. They are an essential resource for examiners who receive training in the correct interpretation and application of the scheme. This training involves, among other things, marking samples of student work and discussing the marks awarded, so as to clarify the correct application of the scheme. The work of examiners is subsequently monitored by Advising Examiners to ensure consistent and accurate application of the marking scheme. This process is overseen by the Chief Examiner, usually assisted by a Chief Advising Examiner. The Chief Examiner is the final authority regarding whether or not the marking scheme has been correctly applied to any piece of candidate work.

Marking schemes are working documents. While a draft marking scheme is prepared in advance of the examination, the scheme is not finalised until examiners have applied it to candidates' work and the feedback from all examiners has been collated and considered in light of the full range of responses of candidates, the overall level of difficulty of the examination and the need to maintain consistency in standards from year to year. This published document contains the finalised scheme, as it was applied to all candidates' work.

In the case of marking schemes that include model solutions or answers, it should be noted that these are not intended to be exhaustive. Variations and alternatives may also be acceptable. Examiners must consider all answers on their merits, and will have consulted with their Advising Examiners when in doubt.

Future Marking Schemes

Assumptions about future marking schemes on the basis of past schemes should be avoided. While the underlying assessment principles remain the same, the details of the marking of a particular type of question may change in the context of the contribution of that question to the overall examination in a given year. The Chief Examiner in any given year has the responsibility to determine how best to ensure the fair and accurate assessment of candidates' work and to ensure consistency in the standard of the assessment from year to year. Accordingly, aspects of the structure, detail and application of the marking scheme for a particular examination are subject to change from one year to the next without notice.

General Guidelines

1. Penalties of three types are applied to candidates' work as follows:

Slips	- numerical slips	(-1)
Blunders	- mathematical errors	(-3)
Misreading	- if not serious	(-1)

Serious blunder or omission or misreading which oversimplifies:
- award the attempt mark only.

Attempt marks are awarded as follows: (A_2)

2. The marking scheme shows one correct solution to each question.
In many cases there are other equally valid methods.

3. Examiners are expected to annotate parts of the responses as directed at the marking conference. (See below.)

Symbol	Name	Use
✗	Cross	Incorrect element
✓	Tick	Correct element (0 marks)
A2	Attempt-2	2 marks
3	3	3 marks
4	4	4 marks
✓ 5	Tick-5	Correct element (5 marks)
	Horizontal wavy line	To be noticed
	Vertical wavy line	Additional page

4. Bonus marks at the rate of 5% of the marks obtained will be given to a candidate who answers entirely through Irish and who obtains 75% or less of the total mark available (i.e. 187 marks or less). In calculating the bonus to be applied decimals are always rounded down, not up ¬ e.g., 4.5 becomes 4; 4.9 becomes 4, etc. See below for when a candidate is awarded more than 187 marks.

Marcanna Breise as ucht freagairt trí Ghaeilge

Léiríonn an tábla thíos an méid marcanna breise ba chóir a bhronnadh ar iarrthóirí a ghnóthaíonn níos mó ná 75% d'iomlán na marcanna.

N.B. Ba chóir marcanna de réir an ghnáthráta a bhronnadh ar iarrthóirí nach ngnóthaíonn níos mó ná 75% d'iomlán na marcanna don scrúdú. Ba chóir freisin an marc bónais sin **a shlánú síos**.

Tábla 250 @ 5%
Bain úsáid as an tábla seo i gcás na n-ábhar a bhfuil 250 marc san iomlán ag gabháil leo agus inarb é 5% gnáthráta an bhónais.

Bain úsáid as an ngnáthráta i gcás 187 marc agus faoina bhun sin. Os cionn an mharc sin, féach an tábla thíos.

Bunmharc	Marc Bónais
188 - 190	9
191 - 196	8
197 - 203	7
204 - 210	6
211 - 216	5

Bunmharc	Marc Bónais
217 - 223	4
224 - 230	3
231 - 236	2
237 - 243	1
244 - 250	0

1. **(a)** A train takes 40 minutes to travel from rest at station A to rest at station B. The distance between the stations is 20 km. The train left station A at 10:00. At 10:15 the speed of the train was 32 km h^{-1} and at 10:30 the speed was 48 km h^{-1}.

The speed of 48 km h^{-1} was maintained until the brakes were applied, causing a uniform deceleration which brought the train to rest at B.

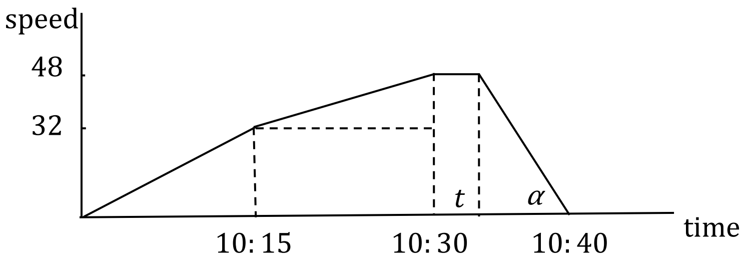
During the first and second 15-minute intervals the accelerations were constant.

(i) Draw a speed-time graph of the motion.

(ii) Find the time taken for the first 16 km.

(iii) Find the deceleration of the train.

(i)



(5)

(ii) $\frac{1}{2} \times \frac{1}{4} \times 32 + \frac{1}{4} \times 32 + \frac{1}{2} \times \frac{1}{4} \times 16 = 14$ (5)

$14 + 48\left(\frac{t_1}{60}\right) = 16$

$t_1 = 2.5$ minutes

time = 32.5 minutes (5)

(iii) $14 + 48\left(\frac{t}{60}\right) + \frac{1}{2} \times \left(\frac{10-t}{60}\right) \times 48 = 20$

$\frac{4}{5}t + \frac{2}{5}(10 - t) = 6$

$t = 5$ min (5)

deceleration = $\tan \alpha = 48 \div \frac{5}{60}$

$= 576 \text{ km h}^{-2} = \frac{2}{45} \text{ m s}^{-2}$ (5) (25)

1. (**b**) A ball *E* is thrown vertically upwards with a speed of 42 m s^{-1}.
T (< 8) seconds later another ball, *F*, is thrown vertically upwards from the same point with the same initial speed.

(i) Find where ball *E* is after 5 s and the total distance it has travelled in this time.

(ii) Prove that when *E* and *F* collide, they will each be travelling with speed $\frac{1}{2}gT$.

(i)

$$s = ut + \tfrac{1}{2}at^2$$
$$s = 42 \times 5 + \tfrac{1}{2} \times (-9.8) \times 5^2$$
$$s = 87.5 \text{ m} \qquad (5)$$

$$v^2 = u^2 + 2as$$
$$0 = 42^2 + 2(-9.8)h$$
$$h = 90 \text{ m}$$

$$d = 90 + (90 - 87.5) = 92.5 \text{ m} \qquad (5)$$

(ii)

$$v_E = 42 - g(t_1)$$
$$v_F = 42 - g(t_1 - T) \qquad (5)$$

$$-v_E = v_F$$
$$-42 + gt_1 = 42 - gt_1 + gT \qquad (5)$$
$$2gt_1 = 84 + gT$$
$$t_1 = \frac{84+gT}{2g}$$

$$v_E = 42 - g\left(\frac{84+gT}{2g}\right)$$
$$v_E = -\tfrac{1}{2}gT$$

$$|v_E| = |v_F| = \tfrac{1}{2}gT \qquad (5) \qquad (25)$$

2. **(a)** A ship is travelling at 22 km h^{-1} in a direction west 30° north. A boat sets out to intercept the ship from a point 25 km south of the ship.
The speed of the boat is 55 km h^{-1}. Find

(i) the direction the boat should steer

(ii) the time, to the nearest minute, that it takes the boat to intercept the ship

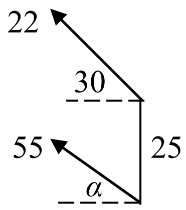
(iii) the distance between the boat and the ship 10 minutes before they meet.

(i) $\vec{V}_{BS} = \vec{V}_B - \vec{V}_S$

$\vec{V}_{BS} = (-55\cos\alpha\ \ \vec{i} + 55\sin\alpha\ \ \vec{j}) - (-22\cos 30\ \ \vec{i} + 22\sin 30\ \ \vec{j})$

$\vec{V}_{BS} = (22\cos 30 - 55\cos\alpha)\ \ \vec{i} + (55\sin\alpha - 22\sin 30)\ \vec{j}$

$\vec{V}_{BS} = 0\ \ \vec{i} + b\ \vec{j}$



$22\cos 30 - 55\cos\alpha = 0$ (5)

$\cos\alpha = \frac{\sqrt{3}}{5} = 0.3464$

$\alpha = 69.73°$ (5)

west 69.73° north

(ii) $b = 55\sin\alpha - 22\sin 30 = 40.594$ (5)

$t = \frac{25}{b} = \frac{25}{40.594} = 0.62\ \text{ h} = 37\text{ minutes}$ (5)

(iii) $d = 40.59 \times \frac{10}{60} = 6.8\ \text{ km}$ (5) (25)

2. **(b)** A woman can swim at u m s^{-1} in still water. In a river she can cover a distance d m against the current in time t_1 and the same distance with the current in time t_2. The current flows parallel to the straight banks at v m s^{-1}.

(i) Show that $v = \frac{d(t_1 - t_2)}{2t_1 t_2}$.

The width of the river is d m and $v < u$.

(ii) Find, in terms of t_1 and t_2, the time taken by the woman to cross the river by the shortest path.

(i)

upstream: $d = (u - v)t_1$ (5)

$$u - v = \frac{d}{t_1}$$

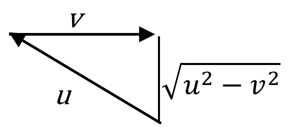
downstream: $d = (u + v)t_2$ (5)

$$u + v = \frac{d}{t_2}$$

$$\Longrightarrow 2v = \frac{d}{t_2} - \frac{d}{t_1}$$

$$v = \frac{d(t_1 - t_2)}{2t_1 t_2}$$ (5)

(ii)



$$\text{time} = \frac{d}{\sqrt{u^2 - v^2}}$$ (5)

$$(u + v)(u - v) = \frac{d}{t_2}\frac{d}{t_1}$$

$$\sqrt{u^2 - v^2} = \frac{d}{\sqrt{t_1 t_2}}$$

$$\text{time} = \frac{d}{\sqrt{u^2 - v^2}} = \sqrt{t_1 t_2}$$ (5) (25)

3. **(a)** A particle is projected out to sea from a point *P* on a cliff to hit a target 60 m horizontally from *P* and 60 m vertically below *P*.

The velocity of projection is $14\sqrt{3}$ m s^{-1} at an angle α to the horizontal.

Find

(i) the two possible values of α

(ii) the times of flight.

(i)

$$14\sqrt{3}\cos\alpha \times t = 60 \qquad (5)$$

$$t = \frac{60}{14\sqrt{3}\cos\alpha}$$

$$14\sqrt{3}\sin\alpha \times t - \frac{1}{2}gt^2 = -60 \qquad (5)$$

$$14\sqrt{3}\sin\alpha \times \left(\frac{60}{14\sqrt{3}\cos\alpha}\right) - \frac{1}{2}g\left(\frac{60}{14\sqrt{3}\cos\alpha}\right)^2 = -60 \qquad (5)$$

$$60\tan\alpha - \frac{1}{2}g\frac{3600}{588\cos^2\alpha} = -60$$

$$60\tan\alpha - 30\left(1+\tan^2\alpha\right) = -60$$

$$\tan^2\alpha - 2\tan\alpha - 1 = 0$$

$$\tan\alpha = 1 \pm \sqrt{2}$$

$$\alpha = -22.5° \text{ or } 67.5° \qquad (5)$$

(ii)

$$t_1 = \frac{60}{14\sqrt{3}\cos 67.5} = 6.47 \text{ s}$$

$$t_2 = \frac{60}{14\sqrt{3}\cos(-22.5)} = 2.68 \text{ s} \qquad (5) \qquad (25)$$

3. **(b)** A particle is projected up a plane with speed u m s^{-1} at an angle β to the plane. The plane is inclined at 30° to the horizontal.

The plane of projection is vertical and contains the line of greatest slope.

Find the greatest range up the plane in terms of u.

$$r_j = 0 \qquad (5)$$

$$u\sin\beta \times t - \tfrac{1}{2} g\cos 30 \times t^2 = 0 \qquad (5)$$

$$t = \frac{2u\sin\beta}{g\cos 30} = \frac{4u\sin\beta}{g\sqrt{3}}$$

$$R = u\cos\beta \times t - \tfrac{1}{2} g\sin 30 \times t^2 \qquad (5)$$

$$= u\cos\beta \times \tfrac{4u\sin\beta}{g\sqrt{3}} - \tfrac{1}{2} g\sin 30 \times \left(\tfrac{4u\sin\beta}{g\sqrt{3}}\right)^2 \qquad (5)$$

$$= \tfrac{2u^2}{g\sqrt{3}} \times \sin 2\beta - \tfrac{4u^2}{3g} \times \sin^2\beta$$

$$\frac{dR}{d\beta} = \tfrac{4u^2}{g\sqrt{3}} \times \cos 2\beta - \tfrac{8u^2}{3g} \times \sin\beta \times \cos\beta$$

$$\frac{dR}{d\beta} = \tfrac{4u^2}{g\sqrt{3}} \times \cos 2\beta - \tfrac{4u^2}{3g} \times \sin 2\beta$$

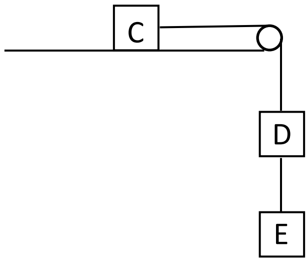
$$\tfrac{dR}{d\beta} = 0 \quad \Rightarrow \quad \tan 2\beta = \sqrt{3}$$

$$\beta = 30°$$

$$R = \tfrac{2u^2}{g\sqrt{3}} \times \sin 60 - \tfrac{4u^2}{3g} \times \sin^2 30$$

$$R = \tfrac{u^2}{g} - \tfrac{u^2}{3g} = \tfrac{2u^2}{3g} \qquad (5) \qquad (25)$$

4. **(a)** A block C of mass $6m$ rests on a rough horizontal table.



It is connected by a light inextensible string which passes over a smooth fixed pulley at the edge of the table to a block D of mass $3m$. D is connected by another light inextensible string to a block E of mass $2m$, as shown in the diagram.

The coefficient of friction between C and the table is $\frac{1}{3}$.

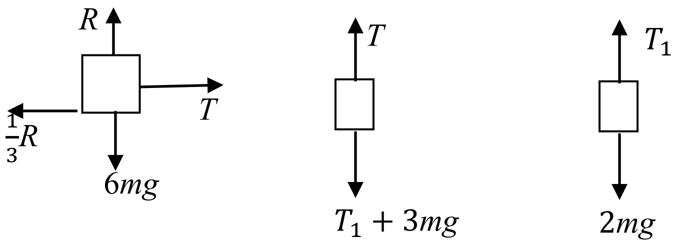
The system is released from rest.

(i) Show on separate diagrams the forces acting on each block.

(ii) Find the acceleration of C.

(iii) Find the tension in each string.

(i)



(5)

(ii)

$$T - 2mg = 6ma \qquad (5)$$

$$T_1 + 3mg - T = 3ma \qquad (5)$$

$$2mg - T_1 = 2ma \qquad (5)$$

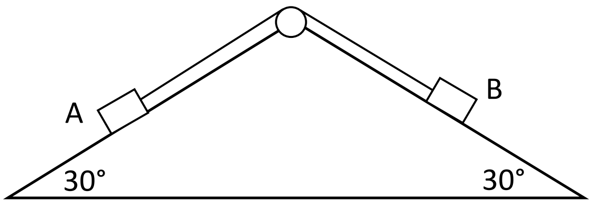
$$a = \frac{3g}{11} \qquad (5)$$

(iii)

$$T = 2mg + 6m \times \frac{3g}{11} \quad \Rightarrow \quad T = \frac{40}{11}mg$$

$$T_1 = 2mg - 2m \times \frac{3g}{11} \quad \Rightarrow \quad T_1 = \frac{16}{11}mg \qquad (5) \qquad (30)$$

4. **(b)** Particles A and B of masses m and $2m$ are connected by a light inextensible string which passes over a pulley at the top of a wedge, one particle resting on each of the faces, which are smooth.

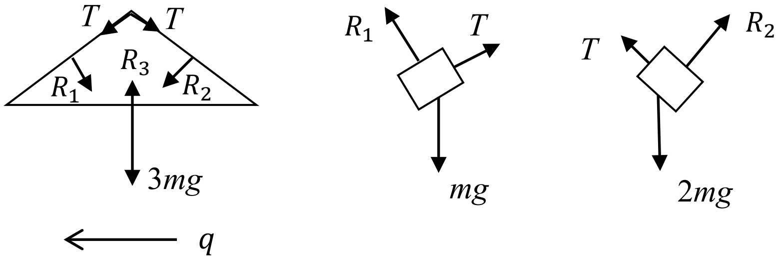


Each of the inclined faces of the wedge makes an angle of 30° with the horizontal.

The wedge of mass $3m$ rests on a smooth horizontal table.

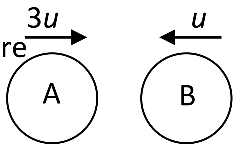
The system is released from rest.

Find the acceleration of the wedge.



$3m$	$R_2 \sin 30 - R_1 \sin 30 = 3mq$	(5)	
	$\Rightarrow \quad R_2 - R_1 = 6mq$		
m	$R_1 - mg\cos 30 = mq \sin 30$	(5)	
$2m$	$2mg\cos 30 - R_2 = 2mq\sin 30$	(5)	
	$mg\cos 30 + R_1 - R_2 = 3mq\sin 30$		
	$mg \times \frac{\sqrt{3}}{2} - 6mq = 3mq \times \frac{1}{2}$		
	$q = \frac{g\sqrt{3}}{15}$	(5)	(20)

5. **(a)** A smooth sphere A of mass $2m$, moving with speed $3u$ on a smooth horizontal table collides directly with a smooth sphere B of mass m, moving in the opposite direction with speed u.



The coefficient of restitution between A and B is e.

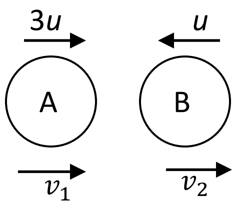
Find, in terms of u and e,

(i) the speed of each sphere after the collision

(ii) the magnitude of the impulse imparted to B due to the collision.

The loss of the kinetic energy due to the collision is $kmu^2(1 - e^2)$.

(iii) Find the value of k.



(i) PCM $\quad 2m(3u) + m(-u) = 2mv_1 + mv_2$ (5)

NEL $\quad v_1 - v_2 = -e\big(3u - (-u)\big)$ (5)

$2v_1 + v_2 = 5u$

$v_1 - v_2 = -4eu$

$v_1 = \frac{u(5-4e)}{3} \qquad v_2 = \frac{u(5+8e)}{3}$ (5), (5)

(ii) $I = \left|m\frac{u(5+8e)}{3} - m(-u)\right|$

$= \frac{8mu}{3}(1+e)$ (5)

(iii) $\mathrm{KE_B} = \frac{1}{2}(2m)(3u)^2 + \frac{1}{2}(m)(-u)^2 = \frac{19}{2}mu^2$

$\mathrm{KE_A} = \frac{1}{2}(2m)(v_1)^2 + \frac{1}{2}(m)(v_2)^2$

$= \frac{1}{9}mu^2\left\{(25 - 40e + 16e^2) + \frac{1}{2}(25 + 80e + 64e^2)\right\}$

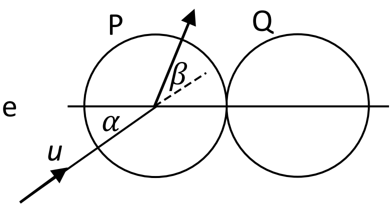
$= \frac{1}{9}mu^2\{37.5 + 48e^2\}$

$\mathrm{KE_L} = \frac{19}{2}mu^2 - \frac{1}{9}mu^2\{37.5 + 48e^2\}$

$= \frac{16}{3}mu^2(1 - e^2)$

$\Rightarrow \quad k = \frac{16}{3}$ (5) (30)

5. **(b)** A smooth sphere P has mass m and speed u. It collides obliquely with a smooth sphere Q, of mass m, which is at rest. Before the collision, the direction of P makes an angle α with the line of centres, as shown in the diagram.



The coefficient of restitution between the spheres is $\frac{1}{3}$.

During the impact the direction of motion of P is turned through an angle β.

Show that $\tan\beta = \frac{2\tan\alpha}{1+3\tan^2\alpha}$.

P	m	$u\cos\alpha\ \vec{i} + u\sin\alpha\ \vec{j}$	$v_1\ \vec{i} + u\sin\alpha\ \vec{j}$
Q	m	$0\ \vec{i} + 0\ \vec{j}$	$v_2\ \vec{i} + 0\ \vec{j}$

PCM $\quad mu\cos\alpha + m(0) = mv_1 + mv_2$ (5)

NEL $\quad v_1 - v_2 = -\frac{1}{3}u\cos\alpha$ (5)

$$v_1 + v_2 = u\cos\alpha$$

$$v_1 - v_2 = -\frac{1}{3}u\cos\alpha$$

$$v_1 = \frac{1}{3}u\cos\alpha$$ (5)

$$\tan(\alpha+\beta) = \frac{u\sin\alpha}{v_1} = 3\tan\alpha$$

$$\frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta} = 3\tan\alpha$$

$$\tan\alpha + \tan\beta = 3\tan\alpha - 3\tan^2\alpha\tan\beta$$

$$\tan\beta = \frac{2\tan\alpha}{1+3\tan^2\alpha}$$ (5) (20)

6. **(a)** A particle moves on a straight line with simple harmonic motion about point *O* as centre. Its displacement from *O* at any time t is x.

At time $t = 0$ the particle passes through a point *H* at a distance of 3 cm from *O*, moving away from *O*. The particle next passes through *H* at time $t = 4$ s, moving towards *O*, and it passes through *H* for a third time after a further 12 s.

(i) Find the period of the motion.

(ii) Show that $x = A\sin(\omega t + \varepsilon)$, where A, ω and ε are constants, satisfies the differential equation

$$\frac{d^2x}{dt^2} = -\omega^2 x.$$

(iii) Find the values of A, ω and ε for the particle.

(i) Period $= 4 + 12 = 16$ s (5)

(ii)

$x = A\sin(\omega t + \varepsilon)$

$\dot{x} = A\omega\cos(\omega t + \varepsilon)$

$\ddot{x} = -A\omega^2\sin(\omega t + \varepsilon) = -\omega^2 x$ (5)

(iii)

$\frac{2\pi}{\omega} = 16$

$\omega = \frac{\pi}{8}$ (5)

$t = 0 \quad \Rightarrow \quad 3 = A\sin\varepsilon$

$t = 4 \quad \Rightarrow \quad 3 = A\sin\left(\frac{\pi}{8} \times 4 + \varepsilon\right) = A\cos\varepsilon$

$\tan\varepsilon = 1$

$\varepsilon = \frac{\pi}{4}$ (5)

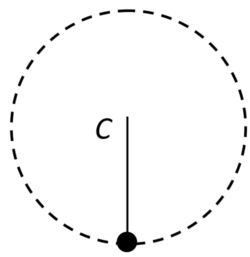
$3 = A\sin\frac{\pi}{4}$

$A = 3\sqrt{2}$ (5) (25)

6. **(b)** A particle is attached to one end of a light inextensible string of length 0.5 m. The other end of the string is attached to a fixed point C. The particle moves in a vertical circle.

The greatest and least tensions in the string are $3T$ and T, respectively.

Find the speed of the particle at the lowest point



$$T + mg = \frac{mv^2}{0.5} \qquad (5)$$

$$3T - mg = \frac{mu^2}{0.5} \qquad (5)$$

$$3T + 3mg = \frac{3mv^2}{0.5}$$

$$-3T + mg = -\frac{mu^2}{0.5}$$

$$4mg = \frac{3mv^2 - mu^2}{0.5}$$

$$v^2 = \frac{u^2 + 2g}{3} \qquad (5)$$

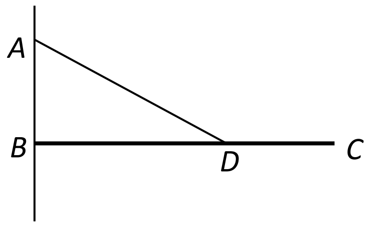
$$\frac{1}{2}mu^2 = \frac{1}{2}mv^2 + mg \times 1 \qquad (5)$$

$$u^2 = v^2 + 2g$$

$$u^2 = \frac{u^2 + 2g}{3} + 2g$$

$$u = 2\sqrt{g} = 6.26 \text{ m s}^{-1} \qquad (5) \qquad (25)$$

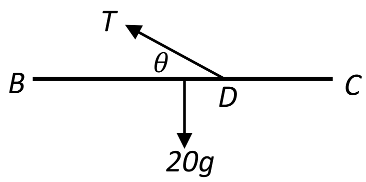
7. **(a)** A uniform rod *BC* of length 3 m, has a mass of 20 kg. The end *B*, about which the rod can turn freely, is attached to a vertical wall. The rod is kept in a horizontal position by a rope attached to a point *D* on the rod and to a point *A* of the wall vertically above *B*, as shown in the diagram.



$|AB| = h$ m and $|BD| = 2$ m.

(i) Prove that the tension in the rope is $\frac{147\sqrt{h^2+4}}{h}$.

(ii) If the tension in the rope cannot exceed 245 N, show that $h \geq 1.5$.



(i) $\circlearrowright B$

$$T\sin\theta \times 2 = 20g \times 1.5 \qquad (5)$$

$$\sin\theta = \frac{h}{\sqrt{h^2+4}} \qquad (5)$$

$$T \times \frac{h}{\sqrt{h^2+4}} \times 2 = 20g \times 1.5$$

$$T = \frac{147\sqrt{h^2+4}}{h} \qquad (5)$$

(ii)

$$T \leq 245$$

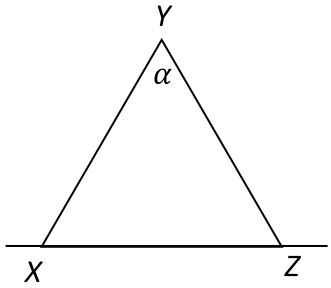
$$\frac{147\sqrt{h^2+4}}{h} \leq 245 \qquad (5)$$

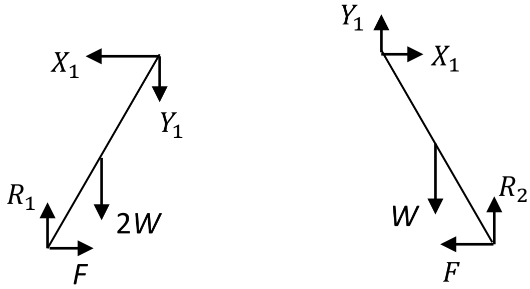
$$\sqrt{h^2+4} \leq \frac{5}{3}h$$

$$16h^2 \geq 36$$

$$h \geq 1.5 \qquad (5) \qquad (25)$$

7. **(b)** Two uniform rods *XY* and *YZ* of equal length and of weights $2W$ and W respectively are smoothly hinged at *Y*.
The rods are at rest in a vertical plane with ends *X* and *Z* on a rough horizontal plane.
$|\angle XYZ| = \alpha$.
If the coefficient of friction is $\frac{\sqrt{3}}{5}$, find the maximum value of α such that the rods remain at rest.





XYZ $\circlearrowright X$

$$R_2 \times 2l\sin\tfrac{1}{2}\alpha = 2W \times \tfrac{1}{2}l\sin\tfrac{1}{2}\alpha + W \times \tfrac{3}{2}l\sin\tfrac{1}{2}\alpha \qquad (5), (5)$$

$$R_2 = \frac{5W}{4} \qquad (5)$$

$$R_1 + R_2 = 3W$$

$$R_1 = \frac{7W}{4}$$

$$R_2 < R_1$$

YZ $\circlearrowright Y$

$$R_2 \times l\sin\tfrac{1}{2}\alpha = F \times l\cos\tfrac{1}{2}\alpha + W \times \tfrac{1}{2}l\sin\tfrac{1}{2}\alpha \qquad (5)$$

$$F = \tfrac{3}{4}\mathrm{W} \times \tan\tfrac{1}{2}\alpha$$

$$\mu R_2 = \tfrac{3}{4}\mathrm{W} \times \tan\tfrac{1}{2}\alpha$$

$$\tfrac{\sqrt{3}}{5} \times \tfrac{5}{4}W = \tfrac{3}{4}W \times \tan\tfrac{1}{2}\alpha$$

$$\tan\tfrac{1}{2}\alpha = \tfrac{1}{\sqrt{3}} \quad \Rightarrow \quad \alpha = 60° \qquad (5) \qquad (25)$$

8. (a) Prove that the moment of inertia of a uniform disc, of mass m and radius r about an axis through its centre, perpendicular to its plane, is $\frac{1}{2}mr^2$.

Let M = mass per unit area

mass of element = $M\{2\pi x dx\}$

moment of inertia of the element = $M\{2\pi x dx\}\, x^2$ (5)

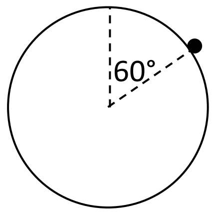
moment of inertia of the disc = $2\pi M \int_0^r x^3\, dx$ (5)

$= 2\pi M \left[\frac{x^4}{4}\right]_0^r$ (5)

$= \frac{1}{2}\pi M r^4$

$= \frac{1}{2}mr^2$ (5) (20)

8. **(b)** A uniform disc of mass 4*m* and radius 20 cm is free to turn about a horizontal axis through its centre perpendicular to its plane.



A particle of mass *m* is attached to the edge of the disc.

Motion starts from the position in which the radius to the particle makes an angle of 60° with the upward vertical.

(i) Find the angular velocity of the disc when the particle is at its lowest point.

(ii) Find the angular displacement of the particle when the angular velocity of the disc is 5 rad s^{-1} for the first time.

(i)

$$I = \frac{1}{2}(4m)\left(\frac{1}{25}\right) + m\left(\frac{1}{25}\right) = \frac{3}{25}m \qquad (5), (5)$$

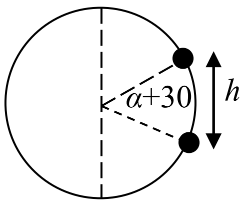
$$h = 0.2 + 0.2\cos 60 = 0.3 \qquad (5)$$

$$\frac{1}{2}I\omega^2 = mgh$$

$$\frac{1}{2}\left(\frac{3}{25}m\right)\omega^2 = mg(0.3) \qquad (5)$$

$$\omega = 7 \ \text{rad s}^{-1} \qquad (5)$$

(ii)



$$\frac{1}{2}\left(\frac{3}{25}m\right)\omega^2 = mgh$$

$$\frac{3}{50}m \times 25 = mgh$$

$$h = \frac{3}{19.6} = \frac{15}{98} \ \text{m} \ = \ 15.31 \text{ cm}$$

$$\sin\alpha = \frac{15.31-10}{20} \Rightarrow \alpha = 15.4°$$

$$\alpha + 30 = 45.4° \qquad (5) \qquad (30)$$

9. **(a)** When placed in liquid A, a uniform solid cylinder floats upright with $\frac{2}{3}$ of its volume immersed in the liquid.

When placed in liquid B, the uniform solid cylinder floats upright with $\frac{4}{5}$ of its volume immersed in the liquid.

What fraction of the cylinder's volume is immersed when the cylinder floats upright in a uniform mixture of equal volumes of liquid A and liquid B?

$$B = W$$

$$\frac{\frac{2}{3}WS_A}{S_O} = W$$

$$S_A = \frac{3}{2}S_O \qquad (5)$$

$$\frac{\frac{4}{5}WS_B}{S_O} = W$$

$$S_B = \frac{5}{4}S_O \qquad (5)$$

$$\frac{kWS_M}{S_O} = W \qquad (5)$$

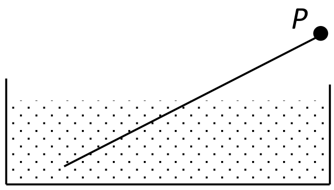
$$S_M = \frac{S_A + S_B}{2} \qquad (5)$$

$$\frac{k}{2}\left(\frac{3}{2}S_O + \frac{5}{4}S_O\right) = S_O$$

$$k = \frac{8}{11} \qquad (5) \qquad (25)$$

9. **(b)** A uniform rod, of length ℓ and weight W, is freely hinged at the point P.

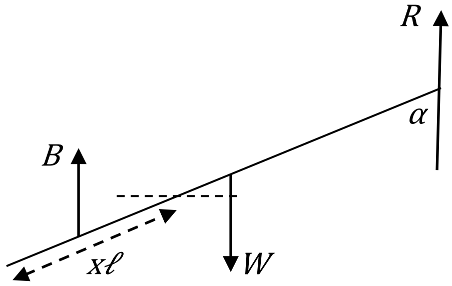
The rod is free to move about a horizontal axis through P.
The other end of the rod is immersed in a liquid of density ρ.
The density of the rod is $s\rho$ ($s < 1$).



The rod is in equilibrium and is inclined as shown in the diagram.
The length of the immersed part of the rod is $x\ell$.

(i) Find x in terms of s.

(ii) If the reaction at the hinge is $\frac{1}{6}W$ upwards, find the value of s.



(i) $B = \frac{xW\rho}{x\rho} = \frac{xW}{s}$ (5)

$B\left\{\ell - \frac{1}{2}x\ell\right\}\sin\alpha = W\frac{1}{2}\ \ell\sin\alpha$ (5)

$\frac{xW}{s}\left\{\ell - \frac{1}{2}x\ell\right\}\sin\alpha = W\frac{1}{2}\ \ell\sin\alpha$

$x\left\{1 - \frac{1}{2}x\right\} = \frac{1}{2}s$

$x^2 - 2x + s = 0$

$x = 1 \pm \sqrt{1-s}$

$x < 1 \quad \Rightarrow \quad x = 1 - \sqrt{1-s}$ (5)

(ii) $R + B = W$ (5)

$\frac{1}{6}W + \frac{x}{s}W = W$

$x = \frac{5}{6}s \quad \Rightarrow \quad 1 - \sqrt{1-s} = \frac{5}{6}s$

$s = \frac{24}{25}$ (5) (25)

10. (a) A particle moves in a horizontal line such that its speed v at time t is given by the differential equation

$$\frac{dv}{dt} = 5 - 8e^{-t}.$$

(i) Given that $v = 2$ when $t = 0$, find an expression for v in terms of t.

(ii) Find the minimum value of v.

(iii) Find the distance travelled by the particle before it attains its minimum speed.

(i)

$$\frac{dv}{dt} = 5 - 8e^{-t}$$

$$\int dv = \int (5 - 8e^{-t})dt \qquad (5)$$

$$[v]_2^v = [5t + 8e^{-t}]_0^t \qquad (5)$$

$$v - 2 = (5t + 8e^{-t}) - 8$$

$$v = 5t + 8e^{-t} - 6 \qquad (5)$$

(ii)

$$\frac{dv}{dt} = 0$$

$$5 - 8e^{-t} = 0$$

$$t = \ln\frac{8}{5} = 0.47$$

$$v_{\min} = 5 \times 0.47 + 5 - 6 = 1.35 \qquad (5)$$

(iii)

$$\frac{ds}{dt} = 5t + 8e^{-t} - 6$$

$$[s]_0^s = \left[\frac{5}{2}t^2 - 8e^{-t} - 6t\right]_0^{0.47}$$

$$s = \left(\frac{5}{2}(0.47)^2 - 8e^{-0.47} - 6(0.47)\right) - (-8)$$

$$s = 0.73 \qquad (5) \qquad (25)$$

10. (b) The rate of decay at any instant of a radioactive substance is proportional to the amount of the substance remaining at that instant. The initial amount of the radioactive substance is N and the amount remaining after time t (hours) is x.

(i) Prove that $x = Ne^{-kt}$, where k is a constant.

(ii) If the initial amount N was reduced to $\frac{N}{3}$ in 14 hours, find the value of k.

(iii) If the amount remaining is reduced from $\frac{N}{3}$ to $\frac{N}{4}$ in t hours, find the value of t.

(i)

$$\frac{dN}{dt} = -kN$$

$$\int \frac{dN}{N} = -k \int dt$$

$$[\ln N]_N^x = -k[t]_0^t$$

$$\ln \frac{x}{N} = -kt$$

$$x = Ne^{-kt} \qquad (5)$$

(ii)

$$[\ln N]_N^{\frac{1}{3}N} = -k[t]_0^{14} \qquad (5)$$

$$\ln \frac{1}{3} = -14k$$

$$k = 0.0785 \qquad (5)$$

(iii)

$$[\ln N]_{\frac{1}{3}N}^{\frac{1}{4}N} = -0.0785[t]_0^t \qquad (5)$$

$$\ln \frac{3}{4} = -0.0785t$$

$$t = 3.7 \qquad (5) \qquad (25)$$