Coimisiún na Scrúduithe Stáit

State Examinations Commission

Leaving Certificate 2021

Marking Scheme

Applied Mathematics

Higher Level

Note to teachers and students on the use of published marking schemes

Marking schemes published by the State Examinations Commission are not intended to be standalone documents. They are an essential resource for examiners who receive training in the correct interpretation and application of the scheme. This training involves, among other things, marking samples of student work and discussing the marks awarded, so as to clarify the correct application of the scheme. The work of examiners is subsequently monitored by Advising Examiners to ensure consistent and accurate application of the marking scheme. This process is overseen by the Chief Examiner, usually assisted by a Chief Advising Examiner. The Chief Examiner is the final authority regarding whether or not the marking scheme has been correctly applied to any piece of candidate work.

Marking schemes are working documents. While a draft marking scheme is prepared in advance of the examination, the scheme is not finalised until examiners have applied it to candidates' work and the feedback from all examiners has been collated and considered in light of the full range of responses of candidates, the overall level of difficulty of the examination and the need to maintain consistency in standards from year to year. This published document contains the finalised scheme, as it was applied to all candidates' work.

In the case of marking schemes that include model solutions or answers, it should be noted that these are not intended to be exhaustive. Variations and alternatives may also be acceptable. Examiners must consider all answers on their merits, and will have consulted with their Advising Examiners when in doubt.

Future Marking Schemes

Assumptions about future marking schemes on the basis of past schemes should be avoided. While the underlying assessment principles remain the same, the details of the marking of a particular type of question may change in the context of the contribution of that question to the overall examination in a given year. The Chief Examiner in any given year has the responsibility to determine how best to ensure the fair and accurate assessment of candidates' work and to ensure consistency in the standard of the assessment from year to year. Accordingly, aspects of the structure, detail and application of the marking scheme for a particular examination are subject to change from one year to the next without notice.

General Guidelines

1 Penalties of three types are applied to candidates' work as follows:

Slips	- numerical slips	S(-1)
Blunders	- mathematical errors	B(-3)
Misreading	- if not serious	M(-1)

Serious blunder or omission or misreading which oversimplifies:
- award the attempt mark only.

Attempt marks are awarded as follows: 5 (att 2)

2 The marking scheme shows one correct solution to each question.
In many cases there are other equally valid methods.

1. **(a)** A ball is thrown vertically downwards from the top of a building of height *h* m. The ball passes the top half of the building in 1.2 s and takes a further 0.8 s to reach the bottom of the building.

Find

(i) the value of *h*

(ii) the speed of the ball at the bottom of the building.

(i)

$$s = ut + \frac{1}{2}at^2$$
$$\frac{1}{2}h = 1.2u + \frac{1}{2}g \times 1.44$$
$$h = 2.4u + 1.44g \qquad (5)$$

$$s = ut + \frac{1}{2}at^2$$
$$h = 2u + \frac{1}{2}g \times 4$$
$$h = 2u + 2g \qquad (5)$$

$$5h = 12u + 70.56$$
$$6h = 12u + 117.6$$

$$h = 47.04 \qquad (5)$$

(ii)

$$h = 2u + 2g$$
$$47.04 = 2u + 19.6$$
$$u = 13.72 \qquad (5)$$

$$v = u + at$$
$$v = 13.72 + 9.8 \times 2$$

$$v = 33.32 \text{ m s}^{-1} \qquad (5) \qquad (25)$$

1. **(b)** Car C, moving with uniform acceleration f passes a point P with speed u (> 0). Two seconds later car D, moving in the same direction with uniform acceleration $2f$ passes P with speed $\frac{6}{5}u$. C and D pass a point Q together. The speeds of C and D at Q are 6.5 m s^{-1} and 9 m s^{-1} respectively.

(i) Show that C travels from P to Q in $(\frac{3}{2f} + 5)$ seconds.

(ii) Find the value of f.

(i) C

$$v = u + at$$
$$6.5 = u + ft \qquad (5)$$

D

$$v = u + at$$
$$9 = \frac{6}{5}u + 2f(t - 2) \qquad (5)$$

$$9 = \frac{6}{5}(6.5 - ft) + 2f(t - 2)$$
$$45 = 39 - 6ft + 10ft - 20f$$
$$4ft = 6 + 20f$$

$$t = \frac{3}{2f} + 5 \qquad (5)$$

(ii) C

$$6.5^2 = u^2 + 2fs$$
$$84.5 = 2u^2 + 4fs$$

D

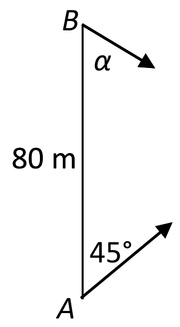
$$9^2 = (1.2u)^2 + 2(2f)s$$
$$81 = 1.44u^2 + 4fs \qquad (5)$$

$$3.5 = 0.56u^2$$
$$u = 2.5$$

$$6.5 = u + ft = u + f\left\{\frac{3}{2f} + 5\right\}$$
$$5 = u + 5f$$
$$5 = 2.5 + 5f$$

$$f = \frac{1}{2} \qquad (5) \qquad (25)$$

2. **(a)** A point B is 80 m north of a point A on a horizontal field. Alan is at point A of the field and Brian is at point B of the field. Alan starts to run in a straight line in the direction north 45° east at a constant speed of 2.5 m s^{-1}.

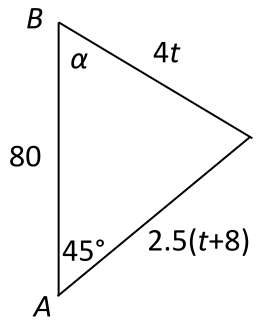


Brian sees Alan start to run, waits 8 seconds, and then runs from B to intercept Alan. Brian runs in a straight line in the direction south α° east at a constant speed of 4 m s^{-1} and intercepts Alan after t seconds.

Find

(i) the value of t

(ii) the value of α.



(i) $$(4t)^2 = 80^2 + \{2.5(t+8)\}^2 - 2 \times 80 \times 2.5(t+8)\cos 45 \qquad (5)$$

$$9.75t^2 + 182.84t - 4537.26 = 0$$

$$t = 14.15 \text{ s} \qquad (5)$$

(ii) $$\frac{\sin\alpha}{2.5(t+8)} = \frac{\sin 45}{4t} \qquad (5)$$

$$\frac{\sin\alpha}{2.5(14.15+8)} = \frac{\sin 45}{4\times 14.15}$$

$$\sin\alpha = \frac{55.375\times\sin 45}{56.6} = 0.6918$$

$$\alpha = \sin^{-1}(0.6918) = 43.77° \qquad (5) \qquad (20)$$

(b) Three aircraft, P, Q and R, are flying at the same height.
P is travelling north at 450 km h^{-1}. Q is travelling at $400\sqrt{2}$ km h^{-1} in a direction east 45° north.
R appears to the pilot of P to be flying in a direction east 10° south.
R appears to the pilot of Q to be flying in a direction east 15.67° south.

Find the magnitude and direction of the velocity of R.

$$\left.\begin{aligned}\overrightarrow{V_{\mathrm{P}}} &= 0\,\vec{\imath} + 450\,\vec{\jmath}\\ \overrightarrow{V_{\mathrm{R}}} &= x\,\vec{\imath} + y\,\vec{\jmath}\end{aligned}\right\}\qquad (5)$$

$$\overrightarrow{V_{\mathrm{RP}}} = x\,\vec{\imath} + (y-450)\,\vec{\jmath} \qquad (5)$$

$$\left.\begin{aligned}\overrightarrow{V_{\mathrm{Q}}} &= 400\,\vec{\imath} + 400\,\vec{\jmath}\\ \overrightarrow{V_{\mathrm{R}}} &= x\,\vec{\imath} + y\,\vec{\jmath}\end{aligned}\right\}\qquad (5)$$

$$\overrightarrow{V_{\mathrm{RQ}}} = (x-400)\,\vec{\imath} + (y-400)\,\vec{\jmath} \qquad (5)$$

$$\tan 10 = \frac{-(y-450)}{x}$$

$$0.1763x + y = 450$$

$$\tan 15.67 = \frac{-(y-400)}{x-400}$$

$$0.2805x + y = 512.2$$

$$0.1042x = 62.2$$

$$x = 596.93 \quad \Rightarrow \quad y = 344.76 \qquad (5)$$

$$\overrightarrow{V_{\mathrm{R}}} = 596.93\,\vec{\imath} + 344.76\,\vec{\jmath}$$

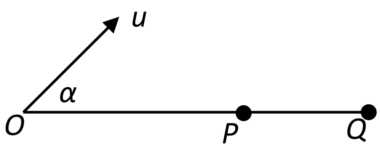
$$\left|\overrightarrow{V_{\mathrm{R}}}\right| = \sqrt{596.93^2 + 344.76^2} = 689.3 \text{ km h}^{-1}$$

$$\tan^{-1}\left(\frac{344.76}{596.93}\right) = \text{east } 30.0° \text{ north} \qquad (5) \qquad (30)$$

3. **(a)** A particle is projected from a point O with speed u m s^{-1} at an angle α to the horizontal.

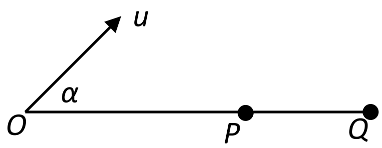
(i) Show that the range of the particle is $\frac{u^2 \sin 2\alpha}{g}$,

and that the maximum range $|OQ|$ is $\frac{u^2}{g}$.



If the angle of projection is increased to 60° the particle strikes the horizontal plane at P.

(ii) Find the distance $|PQ|$ in terms of u.



(i)

$r_j = 0$ (5)

$u \sin\alpha \times t - \frac{1}{2}gt^2$

$t = \frac{2u \sin\alpha}{g}$ (5)

$$\text{Range} = u\cos\alpha \times t$$
$$= u\cos\alpha \times \frac{2u\sin\alpha}{g}$$
$$= \frac{u^2 \sin 2\alpha}{g} \quad (5)$$

$|OQ| = \frac{u^2 \times 1}{g} = \frac{u^2}{g}$ (5)

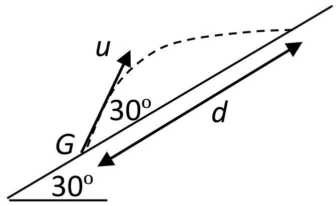
(ii)

$|OP| = \frac{u^2 \times \sin 120}{g} = \frac{u^2\sqrt{3}}{2g}$

$|PQ| = \frac{u^2}{g} - \frac{u^2\sqrt{3}}{2g}$

$= \frac{0.134u^2}{g}$ **or** $0.014u^2$ **or** $\frac{(2-\sqrt{3})u^2}{2g}$ (5) (25)

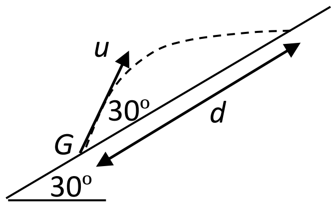
(b) A plane is inclined at an angle of 30° to the horizontal.
A particle is projected from a point G up the plane with initial speed u m s^{-1} at an angle of 30° to the inclined plane.
The range along the inclined plane is d.
The plane of projection is vertical and contains the line of greatest slope.



(i) Find d in terms of u.

When the particle strikes the inclined plane, it bounces vertically upwards.
The coefficient of restitution between the particle and the inclined plane is e.

(ii) Find the value of e.



(i)

$$r_j = 0 \qquad (5)$$

$$u \sin 30 \times t - \tfrac{1}{2} g \cos 30 \ \times t^2 = 0$$

$$t = \frac{2u}{g\sqrt{3}} \qquad (5)$$

$$d = u \cos 30 \times t - \tfrac{1}{2} g \sin 30 \ \times t^2$$

$$= u \times \frac{\sqrt{3}}{2} \times \frac{2u}{g\sqrt{3}} - \frac{1}{2} g \times \frac{1}{2} \times \left(\frac{2u}{g\sqrt{3}}\right)^2$$

$$= \frac{u^2}{g} - \frac{u^2}{3g}$$

$$= \frac{2u^2}{3g} \ \ \textbf{or} \ \ \frac{10u^2}{147} \qquad (5)$$

(ii)

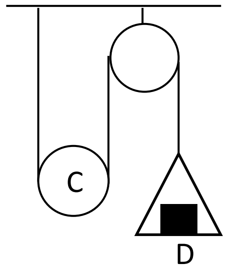
$$\left.\begin{aligned} v_i &= u \cos 30 - g \ \sin 30 \times \frac{2u}{g\sqrt{3}} = \frac{u\sqrt{3}}{6} \\ v_j &= u \sin 30 - g \ \cos 30 \times \frac{2u}{g\sqrt{3}} = -\frac{u}{2} \end{aligned}\right\} \qquad (5)$$

$$\text{Rebound velocity} = \frac{u\sqrt{3}}{6}\ \vec{\imath} + \frac{eu}{2}\ \vec{\jmath}$$

$$\tan 60 = \frac{eu}{2} \div \frac{u\sqrt{3}}{6} = e\sqrt{3}$$

$$e = 1 \qquad (5) \qquad (25)$$

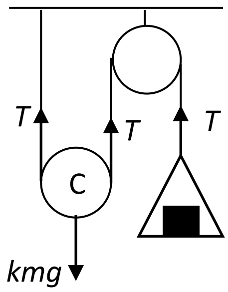
4. **(a)** The diagram shows a light inextensible string having one end fixed, passing under a smooth movable pulley C of mass *km* kg and then over a fixed smooth pulley. The other end of the string is attached to a light scale pan. A bock D of mass *m* kg is placed symmetrically on the centre of the scale pan.
The system is released from rest. The scale pan moves upwards.



(i) Show that $k > 2$.

(ii) Find, in terms of k and m, the tension in the string.

(iii) Find, in terms of k and m, the reaction between D and the scale pan.



(i) $kmg - 2T = kma$ (5)

$T - mg = m \times 2a$ (5)

$a = \left(\frac{k-2}{k+4}\right) g$

$a > 0 \quad \Rightarrow \quad k > 2$ (5)

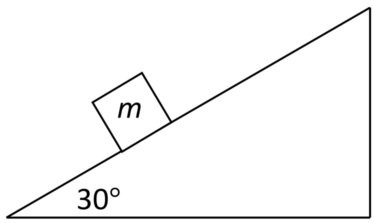
(ii) $T = mg + 2m\left(\frac{k-2}{k+4}\right) g$

$T = \left(\frac{3k}{k+4}\right) mg$ (5)

(iii) $R - mg = m \times 2a$

$R = \left(\frac{3k}{k+4}\right) mg$ (5) (25)

(b) A smooth wedge of mass $4m$ and slope $30°$ rests on a smooth horizontal surface. A particle of mass m is placed on the smooth inclined face of the wedge and is released from rest. A horizontal force F is applied to the wedge to keep it from moving.



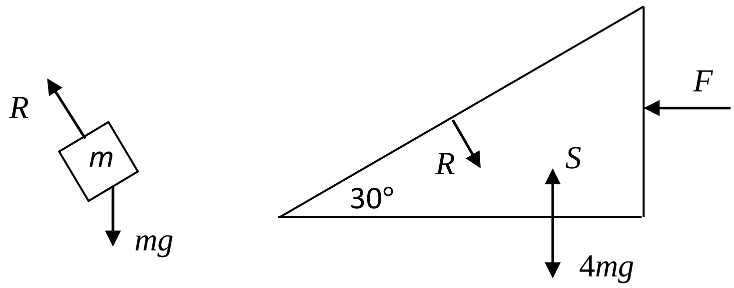
(i) Show, on separate diagrams, the forces acting on the wedge and on the particle.

(ii) Find F in terms of m.

If the force F is removed, the particle moves with acceleration p relative to the wedge and the wedge moves with acceleration q.

(iii) Find the value of p and the value of q.

(i)



(5), (5)

(ii)

$$R = mg\cos 30$$

$$F = R\sin 30$$

$$F = mg\cos 30 \times \sin 30$$

$$F = \frac{\sqrt{3}}{4}mg \qquad (5)$$

(iii)

$$mg\cos 30 - R = mq\sin 30$$

$$R\sin 30 = 4mq$$

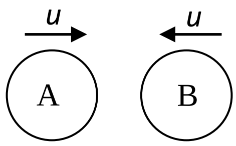
$$\frac{\sqrt{3}}{2}mg - 8mq = \frac{1}{2}mq$$

$$q = \frac{\sqrt{3}}{17}g \qquad (5)$$

$$mg\sin 30 = m(p - q\cos 30)$$

$$p = \frac{10}{17}g \qquad (5) \qquad (25)$$

5. **(a)** A smooth sphere A of mass $4m$, moving with speed u on a smooth horizontal table collides directly with a smooth sphere B of mass m, moving in the opposite direction with speed u.

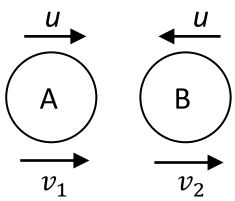


The coefficient of restitution between A and B is e.

(i) Find the speed, in terms of u and e, of each sphere after the collision.

The magnitude of the impulse on B due to the collision is T.

(ii) Show that $\frac{8mu}{5} \leq T \leq \frac{16mu}{5}$.



(i) PCM $\quad 4m(u) + m(-u) = 4mv_1 + mv_2$ (5)

NEL $\quad v_1 - v_2 = -e(u + u)$ (5)

$$4v_1 + v_2 = 3u$$

$$v_1 - v_2 = -2eu$$

$$v_1 = \frac{u(3-2e)}{5} \qquad v_2 = \frac{u(3+8e)}{5}$$ (5), (5)

(ii)

$$T = mv_2 - m(-u)$$

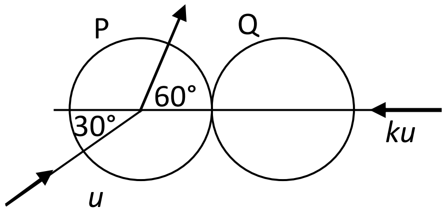
$$T = m\frac{u(3+8e)}{5} - m(-u)$$

$$T = 8mu\frac{(1+e)}{5}$$

$$0 \leq e \leq 1$$

$$\Rightarrow \quad \frac{8}{5}mu \leq T \leq \frac{16}{5}mu$$ (5) (25)

(b) A smooth sphere P has mass $2m$ and speed u. It collides obliquely with a smooth sphere Q of mass m which is moving with speed ku, as shown in the diagram.
Before the collision, the direction of P makes an angle of 30° to the line of centres. After the collision, the direction of P makes an angle of 60° to the line of centres.



The coefficient of restitution between the spheres is e.

(i) Show that $k = \frac{\sqrt{3}(1-e)}{2(1+e)}$.

(ii) Find the speed of Q immediately after the collision.

P	$2m$	$\frac{u\sqrt{3}}{2}\ \vec{\imath} + \frac{u}{2}\ \vec{\jmath}$	$v_1\ \vec{\imath} + \frac{u}{2}\ \vec{\jmath}$
Q	m	$-ku\ \vec{\imath} + 0\ \vec{\jmath}$	$v_2\ \vec{\imath} + 0\ \vec{\jmath}$

(i) PCM $\quad 2m\left(\frac{u\sqrt{3}}{2}\right) + m(-ku) = 2mv_1 + mv_2$ (5)

NEL $\quad v_1 - v_2 = -e\left(\frac{u\sqrt{3}}{2} + ku\right)$ (5)

$$2v_1 + v_2 = u\sqrt{3} - ku$$

$$v_1 - v_2 = -e\left(\frac{u\sqrt{3}}{2}\right) - keu$$

$$6v_1 = 2u\sqrt{3} - 2ku - eu\sqrt{3} - 2keu$$ (5)

$$\tan 60 = \frac{\frac{u}{2}}{v_1}$$

$$v_1 = \frac{u}{2\sqrt{3}} \quad \Rightarrow \quad 6v_1 = u\sqrt{3}$$

$$2u\sqrt{3} - 2ku - eu\sqrt{3} - 2keu = u\sqrt{3}$$

$$k = \frac{\sqrt{3}(1-e)}{2(1+e)}$$ (5)

(ii)

$$3v_2 = u\sqrt{3} - ku + eu\sqrt{3} + 2keu$$

$$3v_2 = u\sqrt{3} - \frac{\sqrt{3}(1-e)}{2(1+e)}u + eu\sqrt{3} + \frac{\sqrt{3}(1-e)}{(1+e)}eu$$

$$3v_2 = u\frac{(\sqrt{3}+7e\sqrt{3})}{2(1+e)}$$

$$v_2 = \frac{u\sqrt{3}(1+7e)}{6(1+e)}$$ (5) (25)

6. **(a)** A particle D of mass *m* is suspended from a fixed point by a light elastic string of natural length ℓ and elastic constant $\frac{3mg}{\ell}$.

Initially D rests in equilibrium with the string vertical.

The particle is now pulled down a vertical distance $\frac{2}{3}\ell$ below its equilibrium position and released from rest.

(i) Show that D moves with simple harmonic motion.

(ii) In terms of ℓ, find the height above the equilibrium position to which D rises.

(i)

$T_0 = mg$ (5)

$ke = mg$

$\frac{3mg}{\ell}e = mg$

$e = \frac{1}{3}\ell$ (5)

$ma = mg - T$

$ma = mg - \frac{3mg}{\ell}(e + x)$

$a = -\frac{3g}{\ell}x$ (5)

(ii)

$\omega = \sqrt{\frac{3g}{\ell}}$

$v = \omega\sqrt{A^2 - x^2}$

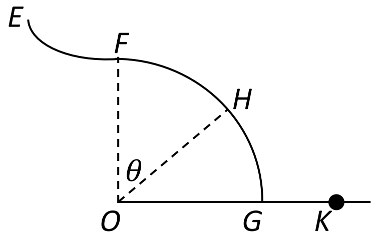
$v = \sqrt{\frac{3g}{\ell}}\sqrt{\left(\frac{2}{3}\ell\right)^2 - \left(\frac{1}{3}\ell\right)^2} = \sqrt{g\ell}$ (5)

$v^2 = u^2 + 2as$

$0 = g\ell - 2gs \quad \Rightarrow \quad s = \frac{1}{2}\ell$

height $= \frac{1}{3}\ell + \frac{1}{2}\ell = \frac{5}{6}\ell$ (5) (25)

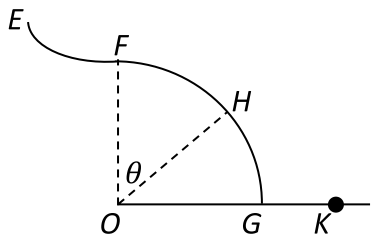
(b) A smooth slide *EFG* is in the shape of two arcs, *EF* and *FG*, each of radius *r*. The centre *O* of arc *FG* is vertically below *F* as shown in the diagram.
Point *E* is at a height $\frac{r}{5}$ above point *F*.
A child starts from rest at *E*, moves along the slide past the point *F* and loses contact with the slide at point *H*.
OH makes an angle θ with the vertical.



(i) Find the value of θ.

The child lands in a sandpit at point *K*.

(ii) Find, in terms of *r*, the speed of the child at *K*.



(i) *H* $\quad \frac{1}{2}mv^2 = mg\left\{\frac{1}{5}r + (r - r\cos\theta)\right\}$ (5)

$$v^2 = 2gr\left\{\frac{1}{5} + (1 - \cos\theta)\right\}$$

$$v^2 = 2gr\left\{\frac{6}{5} - \cos\theta\right\}$$

H $\quad mg\cos\theta - R = \frac{mv^2}{r}$ (5)

$mg\cos\theta - 0 = 2mg\left\{\frac{6}{5} - \cos\theta\right\}$ (5)

$$\cos\theta = \frac{4}{5}$$

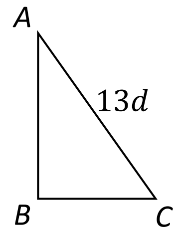
$\Rightarrow \quad \theta = 36.87°$ (5)

(ii) *K* $\quad \frac{1}{2}m{v_1}^2 = mg\left(r + \frac{1}{5}r\right)$

$${v_1}^2 = \frac{12}{5}gr$$

$v_1 = \sqrt{\frac{12}{5}gr}$ **or** $\frac{14\sqrt{3r}}{5}$ **or** $4.85\sqrt{r}$ (5) (25)

7. **(a)** A thin uniform rod, of length $30d$ and mass m, is bent to form a frame. The frame is in the shape of a right-angled triangle *ABC*, as shown in the diagram.
$|CA| = 13d$ and $|AB| > |BC|$.

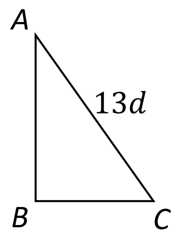


(i) Find $|BC|$ in terms of d.

(ii) Find the distance of the centre of gravity of the frame from *AB*.

The frame is freely suspended from *A*. A horizontal force of magnitude *kmg*, where *k* is a constant, is applied to the frame at *B*. The line of action of the force lies in the vertical plane containing the frame. The frame hangs in equilibrium with *AB* vertical.

(iii) Find the value of *k*.



(i) $|BC| = x \quad \Rightarrow \quad |AB| = 17d - x$

$x^2 + (17d - x)^2 = (13d)^2$

$|BC| = x = 5d$ (5)

(ii) $\circlearrowright AB$

$m_{AC} \times \frac{5}{2}d + m_{BC} \times \frac{5}{2}d = m \times y$ (5)

$\frac{5}{30}m \times \frac{5}{2}d + \frac{13}{30}m \times \frac{5}{2}d = m \times y$

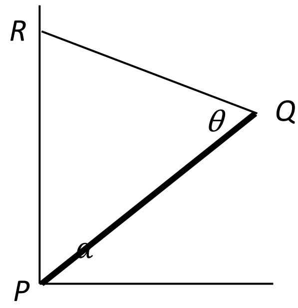
$y = \frac{3}{2}d$ (5)

(iii) $\circlearrowright A$

$kmg \times 12d = mg \times \frac{3}{2}d$ (5)

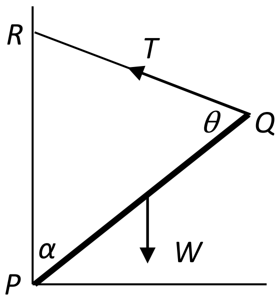
$k = \frac{1}{8}$ (5) (25)

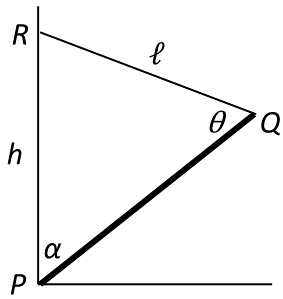
(b) A uniform rod *PQ* of weight *W* rests against the junction of the horizontal ground and a vertical wall.
It is supported by a string of length ℓ attached to *Q* and to a point *R* vertically above *P* on the wall.
The rod makes an angle α with the wall and the string makes an angle θ with the rod, as shown in the diagram.
$|PR| = h$.



(i) Find T, the tension in the string, in terms of *W*, ℓ and h.

(ii) If $T = \frac{1}{3}W$, find ℓ in terms of h.





(i) *PQ* $\circlearrowright P$

$$T\sin\theta \times |PQ| = W \times \tfrac{1}{2}|PQ|\sin\alpha \qquad (5), (5)$$

$$\frac{2T}{W} = \frac{\sin\alpha}{\sin\theta}$$

$$\frac{\sin\alpha}{\ell} = \frac{\sin\theta}{h} \qquad (5)$$

$$\frac{\sin\alpha}{\sin\theta} = \frac{\ell}{h}$$

$$T = \frac{W\ell}{2h} \qquad (5)$$

(ii)

$$T = \frac{W\ell}{2h}$$

$$\frac{W}{3} = \frac{W\ell}{2h}$$

$$\ell = \frac{2}{3}h \qquad (5) \qquad (25)$$

8. (a) Prove that the moment of inertia of a uniform rod, of mass *m* and length 2ℓ about an axis through its centre, perpendicular to its plane, is $\frac{1}{3}m\ell^2$.

mass of element = $M\{dx\}$

moment of inertia of the element = $M\{dx\}\,x^2$ (5)

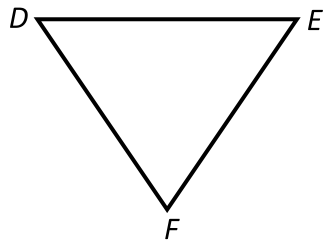
moment of inertia of the rod = $M\int_{-\ell}^{\ell} x^2\,dx$ (5)

$= M\left[\frac{x^3}{3}\right]_{-\ell}^{\ell}$ (5)

$= \frac{2}{3}M\ell^3$

$= \frac{1}{3}m\ell^2$ (5) (20)

(b) Three equal uniform rods, each of mass *m* and length 2ℓ, form the sides of a rigid equilateral triangular frame *DEF*.



The frame is free to rotate in a vertical plane about a fixed smooth horizontal axis which passes through *D* and is perpendicular to the plane of the frame.

(i) Show that the moment of inertia of the frame about the axis is $6m\ell^2$.

The frame is held with *DE* horizontal and *F* below *DE*. It is then released from rest.

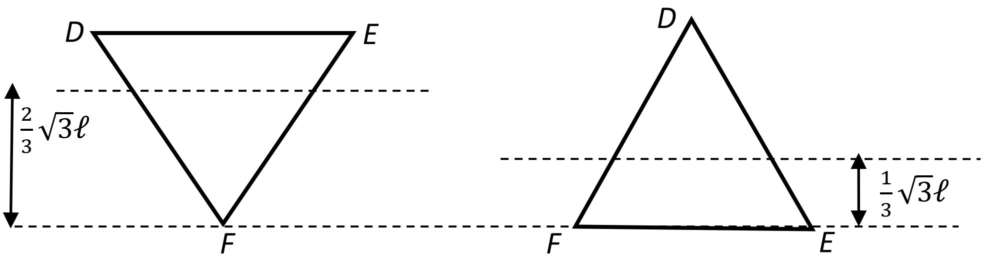
(ii) Find, in terms of ℓ, the angular speed of the frame when *FE* is horizontal for the first time.

(iii) If the period of small oscillations for the frame is 1.87 s, find the value of ℓ.

(i) $I = \frac{4}{3}m\ell^2 + \frac{4}{3}m\ell^2 + \left\{\frac{1}{3}m\ell^2 + m(2\ell\cos 30)^2\right\}$ (5)

$I = 6m\ell^2$ (5)

(ii)



$$Mgh_1 = \frac{1}{2}I\omega^2 + Mgh_2$$

$$3m \times g \times \frac{2}{3}\sqrt{3}\ell = \frac{1}{2}\left(6m\ell^2\right)\omega^2 + 3m \times g \times \frac{1}{3}\sqrt{3}\ell$$ (5), (5)

$$\sqrt{3}mg\ell = 3m\ell^2\omega^2$$

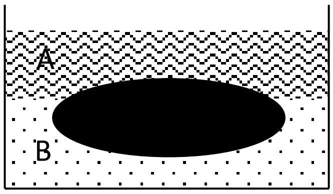
$$\omega = \sqrt{\frac{\sqrt{3}g}{3\ell}}$$ (5)

(iii) $Mgh = mg \times \frac{1}{2}\sqrt{3}\ell + mg \times \frac{1}{2}\sqrt{3}\ell + mg \times \sqrt{3}\ell$

$$T = 1.87 = 2\pi\sqrt{\frac{I}{Mgh}} = 2\pi\sqrt{\frac{6m\ell^2}{2mg\sqrt{3}\ell}} = 2\pi\sqrt{\frac{\sqrt{3}\ell}{g}}$$

$\ell = 0.50$ m. (5) (30)

9. **(a)** Liquid A of relative density 0.8 rests on top of liquid B of relative density 1.4 without mixing. A solid object of relative density 1.2 floats with part of its volume in liquid A and the remainder in liquid B.
Find the fraction of the volume of the object immersed in liquid B.



$B_{\mathrm{A}} = 800 \times (1-k)V \times g$ (5)

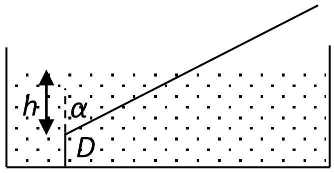
$B_{\mathrm{B}} = 1400 \times kV \times g$ (5)

$W = B_{\mathrm{A}} + B_{\mathrm{B}}$ (5)

$1200Vg = 800(1-k)Vg + 1400kVg$

$k = \frac{2}{3}$ (5) (20)

(b) A uniform rod of length ℓ is tied by an inelastic string at its lower end D to the base of a tank. D is at a depth h below the surface of a liquid.
The relative density of the rod is s.
The relative density of the liquid is ρ.
The rod is in equilibrium, inclined at an angle α to the vertical as shown in the diagram.

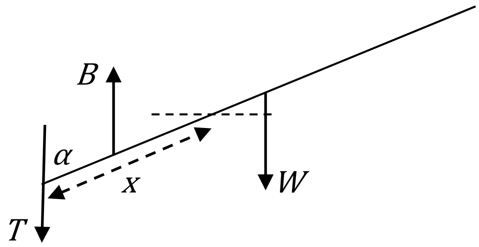


(i) Show that the length of the immersed part of the rod is $\ell\sqrt{\frac{s}{\rho}}$.

If $4\rho h^2 = s\ell^2$ find

(ii) the value of α

(iii) s in terms of ρ, if the magnitude of the tension in the string is $\frac{1}{2}W$.



(i) $$B = \frac{\frac{x}{\ell}W\rho}{s} = \frac{xW\rho}{s\ell}$$ (5)

$$B\left\{\frac{1}{2}x\right\}\sin\alpha = W\frac{1}{2}\,\ell\sin\alpha$$ (5)

$$\frac{xW\rho}{s\ell}\left\{\frac{1}{2}x\right\} = W\frac{1}{2}\,\ell$$

$$x^2\rho = s\ell^2$$

$$x = \ell\sqrt{\frac{s}{\rho}}$$ (5)

(ii) $$h = x\cos\alpha$$ (5)

$$\cos\alpha = \sqrt{\frac{h^2}{x^2}} = \sqrt{\frac{h^2\rho}{\ell^2 s}} = \frac{1}{2}$$

$$\alpha = 60°$$ (5)

(iii) $$h = x\cos 60 = \frac{1}{2}x$$

$$B = \frac{xW\rho}{s\ell} = \frac{2hW\rho\ell}{4h^2\rho} = \frac{\ell}{2h}W$$

$$B = T + W = \frac{3}{2}W$$

$$\frac{\ell}{2h}W = \frac{3}{2}W \quad \Rightarrow \quad \ell = 3h = \frac{3}{2}x$$

$$\sqrt{\frac{\rho}{s}} = \frac{\ell}{x} = \frac{3}{2} \quad \Rightarrow \quad s = \frac{4}{9}\rho$$ (5) (30)

10. (a) A car of mass 1200 kg starts from rest and travels along a straight horizontal road. The engine of the car exerts a constant power of 3000 W.

If there is no resistance to the motion of the car, find

(i) the speed of the car after 3 minutes

(ii) the average speed of the car during this time.

(i)

$$F = \frac{P}{v} = \frac{3000}{v} \qquad (5)$$

$$1200 \times \frac{dv}{dt} = \frac{3000}{v}$$

$$\int v\, dv = \frac{5}{2}\int dt \qquad (5)$$

$$\left[\frac{1}{2}v^2\right]_0^v = \frac{5}{2}[t]_0^{180} \qquad (5)$$

$$\frac{1}{2}v^2 = 450$$

$$v = 30 \text{ m s}^{-1} \qquad (5)$$

(ii)

$$1200 \times v\frac{dv}{ds} = \frac{3000}{v}$$

$$\int v^2\, dv = \frac{5}{2}\int ds$$

$$\left[\frac{1}{3}v^3\right]_0^{30} = \frac{5}{2}[s]_0^s$$

$$s = 3600$$

$$\text{average speed} = \frac{3600}{180} = 20 \text{ m s}^{-1} \qquad (5) \qquad (25)$$

(b) P, the population of insects in a region, grows at a rate that is proportional to the current population.

$$\frac{dP}{dt} = kP$$

where k is a positive constant. In the absence of any outside factors the population will triple in 15 days.

(i) Find the value of k.

A scientist begins to remove 10 insects from the population each day.

(ii) If there are initially 120 insects in the region the population will not survive. After how many days will the population die out?

(i) $\frac{dP}{dt} = kP$

$\int \frac{dP}{P} = k \int dt$ (5)

$[\ln P]_P^{3P} = k[t]_0^{15}$ (5)

$ln3P - lnP = 15k$

$k = \frac{ln3}{15} = 0.07324$ (5)

(ii) $\frac{dP}{dt} = \frac{ln3}{15}P - 10$

$\int \frac{dP}{\frac{ln3}{15}P - 10} = \int dt$

$\frac{15}{ln3}\left[ln\left|\frac{ln3}{15}P - 10\right|\right]_{120}^{0} = [t]_0^t$ (5)

$t = \frac{15}{ln3}(ln10 - ln1.2111)$

$t = 28{\cdot}8$ days (5) (25)