Coimisiún na Scrúduithe Stáit
State Examinations Commission

LEAVING CERTIFICATE 2009

MARKING SCHEME

MATHEMATICS

HIGHER LEVEL

Contents

GENERAL GUIDELINES FOR EXAMINERS – PAPER 1

1. Penalties of three types are applied to candidates' work as follows:
   - Blunders - mathematical errors/omissions (-3)
   - Slips - numerical errors (-1)
   - Misreadings (provided task is not oversimplified) (-1).

   Frequently occurring errors to which these penalties must be applied are listed in the scheme. They are labelled: B1, B2, B3,…, S1, S2,…, M1, M2,…etc. These lists are not exhaustive.

2. When awarding attempt marks, e.g. Att(3), note that
   - any *correct, relevant* step in a part of a question merits at least the attempt mark for that part
   - if deductions result in a mark which is lower than the attempt mark, then the attempt mark must be awarded
   - a mark between zero and the attempt mark is never awarded.

3. Worthless work is awarded zero marks. Some examples of such work are listed in the scheme and they are labelled as W1, W2,…etc.

4. The phrase "hit or miss" means that partial marks are not awarded – the candidate receives all of the relevant marks or none.

5. The phrase "and stops" means that no more work of merit is shown by the candidate.

6. Special notes relating to the marking of a particular part of a question are indicated by an asterisk. These notes immediately follow the box containing the relevant solution.

7. The sample solutions for each question are not intended to be exhaustive lists – there may be other correct solutions. Any examiner unsure of the validity of the approach adopted by a particular candidate to a particular question should contact his/her advising examiner.

8. Unless otherwise indicated in the scheme, accept the best of two or more attempts – even when attempts have been cancelled.

9. The *same* error in the *same* section of a question is penalised *once* only.

10. Particular cases, verifications and answers derived from diagrams (unless requested) qualify for attempt marks at most.

11. A serious blunder, omission or misreading results in the attempt mark at most.

12. Do not penalise the use of a comma for a decimal point, e.g. €5.50 may be written as €5,50.

QUESTION 1

Part (a)	**10 (5, 5) marks**	**Att (2, 2)**
Part (b)	**20 (5, 5, 5, 5) marks**	**Att (2, 2, 2, 2)**
Part (c)	**20 (5, 5, 5, 5) marks**	**Att (2, 2, 2, 2)**

Part (a) **10 (5, 5) marks** **Att (2, 2)**

1. (a) Find the value of $\frac{x}{y}$ when $\frac{2x+3y}{x+6y}=\frac{4}{5}$.

Cross Multiplication	**5 marks**	**Att 2**
Finish	**5 marks**	**Att 2**

1 (a)

$$\frac{2x+3y}{x+6y}=\frac{4}{5} \Rightarrow 10x+15y=4x+24y \Rightarrow 6x=9y. \quad \therefore \quad \frac{x}{y}=\frac{9}{6}=\frac{3}{2}.$$

Blunders (-3)
B1 Incorrect cross multiplication

Slips (-1)
S1 Numerical

S2 $\frac{y}{x}$

OR

Correct Ratio	**5 marks**	**Att 2**
Solving	**5 marks**	**Att 2**

1 (a)

Let numerator = 4 and denominator = 5 (or 8 & 10 respectively, etc.)

$$\Rightarrow (i): 2x+3y=4 \quad \times 2 \Rightarrow 4x+6y=8$$
$$(ii): x+6y=5 \quad \times 1 \Rightarrow \underline{x+6y=5}$$
$$3x=3 \quad \Rightarrow \quad x=1$$

(ii): $x+6y=5$

$$(1)+6y=5$$
$$6y=4 \Rightarrow y=\frac{4}{6}=\frac{2}{3}$$
$$\frac{x}{y}=\frac{1}{\left(\frac{2}{3}\right)}=\frac{3}{2}$$

Blunders (-3)
B1 Error in ratio

B2 No $\frac{x}{y}$

Slips (-1)
S1 Numerical

S2 $\frac{y}{x}$

Part (b) **20 (5, 5, 5, 5) marks** **Att (2, 2, 2, 2)**

(b) Let $f(x) = x^2 - 7x + 12$.

(i) Show that if $f(x+1) \neq 0$, then $\dfrac{f(x)}{f(x+1)}$ simplifies to $\dfrac{x-4}{x-2}$.

(ii) Find the range of values of x for which $\dfrac{f(x)}{f(x+1)} > 3$.

(b) (i) *f*(*x*+1)	**5 marks**	**Att 2**
Simplification	**5 marks**	**Att 2**

1 (b) (i)

$$f(x) = x^2 - 7x + 12 \Rightarrow f(x+1) = (x+1)^2 - 7(x+1) + 12.$$

$$\frac{f(x)}{f(x+1)} = \frac{x^2 - 7x + 12}{x^2 - 5x + 6} = \frac{(x-3)(x-4)}{(x-3)(x-2)} = \frac{x-4}{x-2}.$$

Blunders (-3)

B1 Expansion $(x+1)^2$ once only

B2 Incorrect fraction

B3 Factors

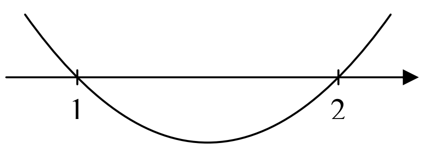
(b) (ii) Quadratic Inequality	**5 marks**	**Att 2**
Range	**5 marks**	**Att 2**

1 (b) (ii)

$$\frac{f(x)}{f(x+1)} > 3 \quad \Rightarrow \frac{x-4}{x-2} > 3$$

Multiply across by $(x-2)^2 > 0$

$$(x-2)(x-4) > 3(x-2)^2$$
$$x^2 - 6x + 8 > 3(x^2 - 4x + 4)$$
$$x^2 - 6x + 8 > 3x^2 - 12x + 12$$
$$0 > 2x^2 - 6x + 4$$
$$0 > x^2 - 3x + 2$$
$$0 > (x-1)(x-2)$$



Range : $1 < x < 2$

Blunders (-3)

B1 Inequality sign

B2 Indices

B3 Expansion of $(x-2)^2$ once only

B4 Factors

B5 Roots formula once only

B6 Deduction root from factor
B7 Range not stated
B8 Incorrect range
B9 Shape graph

Slips (-1)
S1 Numerical

Attempts
A1 Linear inequality only

Worthless
W1 Squares both sides

OR
(When not treated as a quadratic)

(b) (ii) case $(x-2)>0$	**5 marks**	**Att 2**
case $(x-2)<0$	**5 marks**	**Att 2**

1 (b) (ii)

case (a): $x-2>0$ (so $x>2$)

$$\frac{x-4}{x-2}>3$$

$\Leftrightarrow (x-4)>3(x-2)$ since $x-2>0$

$\Leftrightarrow x-4>3x-6$

$\Leftrightarrow 2>2x$

$\Leftrightarrow 1>x$

Not possible when $x>2$ $\Rightarrow$ no solution from this case.

case (b): $x-2<0$ (so $x<2$)

$$\frac{x-4}{x-2}>3$$

$\Leftrightarrow x-4<3(x-2)$ since $x-2<0$

$\Leftrightarrow x-4<3x-6$

$\Leftrightarrow 2<2x$

$\Leftrightarrow 1<x$

$\Rightarrow 1<x<2$

OR

(b) (ii) case $(x-2)>0$	**5 marks**	**Att 2**
case $(x-2)<0$	**5 marks**	**Att 2**

1 (b) (ii)

$$\frac{x-4}{x-2} > 3 \quad \Rightarrow \quad \frac{x-4}{x-2} - 3 > 0$$

$$\frac{(x-4)-3(x-2)}{(x-2)} > 0$$

$$\frac{x-4-3x+6}{(x-2)} > 0$$

$$\frac{-2x+2}{x-2} > 0$$

So, need numerator and denominator to have same sign.

case (a): $x-2>0$ **and** $-2x+2>0$

$x>2$ $\qquad 2>2x$

$\qquad 1>x$

Not possible $\Rightarrow$ no solution from this case.

case (b) $x-2<0$ **and** $-2x+2<0$

$x<2$ $\qquad 2<2x$

$\qquad 1<x$

$$\Rightarrow 1<x<2$$

Blunders (-3)

B1 Inequality sign

B2 Deduction of value

B3 Range not stated

B4 Incorrect range

Slips (-1)

S1 Numerical

Part (c)	20 (5, 5, 5, 5) marks	Att (2, 2, 2, 2)

(c) Given that $x-c+1$ is a factor of $x^2-5x+5cx-6b^2$, express c in terms of b.

Division	**5 marks**	**Att 2**
Remainder = 0	**5 marks**	**Att 2**
Quadratic in *b* and *c*	**5 marks**	**Att 2**
Values of *c*	**5 marks**	**Att 2**

1 (c)

$$\begin{array}{r|l} & x+(-6+6c) \\ x-c+1 & x^2-5x+5cx-6b^2 \\ & \underline{x^2+x-cx} \\ & x(-6+6c)-6b^2 \\ & \underline{x(-6+6c)-c(-6+6c)+(-6+6c)} \\ & -6b^2+c(-6+6c)-(-6+6c) \end{array}$$

$$\therefore \quad -6b^2-6c+6c^2+6-6c=0$$

$$c^2-2c+1=b^2.$$

$$(c-1)^2=b^2 \Rightarrow c-1=\pm b \Rightarrow c=1\pm b.$$

Blunders (-3)

B1 Indices

B2 Not like to like when equation coefficients

B3 Only one value of *c* given

B4 Factors

Slips (-1)

S1 Not changing sign when subtracting

Attempts

A1 Any effort at division

Other linear factor	**5 marks**	**Att 2**
Equating coefficients	**5 marks**	**Att 2**
Quadratic in *b* and *c*	**5 marks**	**Att 2**
Values of *c*	**5 marks**	**Att 2**

1 (c)

$$f(x) = x^2 - 5x + 5cx - 6b^2 = (x - c + 1)\left(x - \frac{6b^2}{-c+1}\right)$$

$$(x + 1 - c)\left(x - \frac{6b^2}{1-c}\right) = x^2 - cx + x - \frac{6b^2 x}{1-c} + \frac{6b^2 c}{1-c} - \frac{6b^2}{1-c}$$

$$= x^2 - x\left(c - 1 + \frac{6b^2}{1-c}\right) + \frac{6b^2 c - 6b^2}{1-c}$$

Equating Coefficients of x:

$$5 - 5c = c - 1 + \frac{6b^2}{1-c}$$

$$6 - 6c = \frac{6b^2}{1-c}$$

$$(1-c) = \frac{b^2}{(1-c)}$$

$$(1-c)^2 = b^2$$

$$1 - c = \pm b$$

$$c = 1 \pm b$$

Blunders (-3)

B1 Indices

B2 Only 1 value of c given

B3 Factors

Root (c-1)	**5 marks**	**Att 2**
$f(c-1)$ substituted	**5 marks**	**Att 2**
Quadratic in b and c	**5 marks**	**Att 2**
Values of c	**5 marks**	**Att 2**

1 (c)

$(x-c+1)$ is a factor of $f(x)$ $\Rightarrow$ $(c-1)$ is a root

$\Rightarrow f(c-1)=0$

$$f(x)=x^2-5x+5cx-6b^2$$

$$f(c-1)=(c-1)^2-5(c-1)+5c(c-1)-6b^2=0$$

$$c^2-2c+1-5c+5+5c^2-5c=6b^2$$

$$6c^2-12c+6 \quad =6b^2$$

$$6(c^2-2c+1) \quad =6(b^2)$$

$$(c-1)^2 \quad =b^2$$

$$c-1 \quad =\pm b$$

$$c=1\pm b$$

Blunders (-3)

B1 Indices

B2 Expansion of $(c-1)^2$ once only

B3 Only 1 value of c given

B4 Factors

QUESTION 2

Part (a)	**10 (5, 5) marks**	**Att (2, 2)**
Part (b)	**20 (10, 5, 5) marks**	**Att (3, 2, 2)**
Part (c)	**20 (5, 5, 5, 5) marks**	**Att (2, 2, 2, 2)**

Part (a)	**10 (5, 5) marks**	**Att (2, 2)**

2. **(a)** Solve the simultaneous equations

$$x - y + 8 = 0$$

$$x^2 + xy + 8 = 0.$$

Quadratic	**5 marks**	**Att 2**
Values	**5 marks**	**Att 2**

2 (a)

$$x = y - 8. \quad \therefore \; (y-8)^2 + y(y-8) + 8 = 0.$$

$$y^2 - 16y + 64 + y^2 - 8y + 8 = 0$$

$$2y^2 - 24y + 72 = 0 \Rightarrow y^2 - 12y + 36 = 0.$$

$$(y-6)^2 = 0 \Rightarrow y = 6.$$

$\therefore$ Solution is $(-2, 6)$.

Blunders (-3)

B1 Indices
B2 Factors once only
B3 Deduction value from factor
B4 Not getting 2nd value (having got 1st)
B5 Roots formula once only

Slips (-1)

S1 Numerical

Attempts

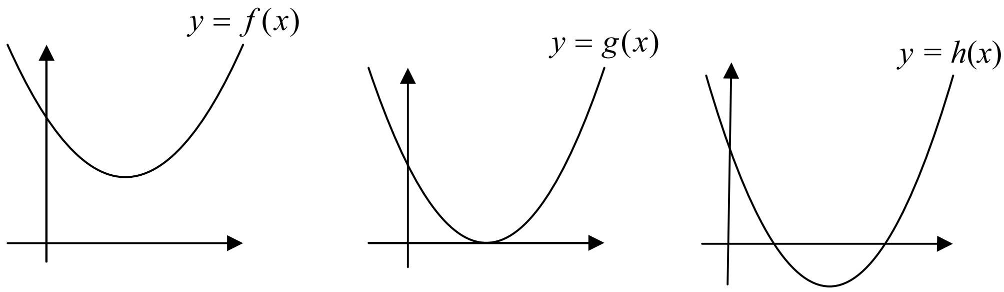
A1 Not quadratic

Worthless

W1 Trial and error

Part (b) **20 (10, 5, 5) marks** **Att (3, 2, 2)**

(b) (i) The graphs of three quadratic functions, *f*, *g* and *h*, are shown.



In each case, state the nature of the roots of the function.

(ii) The equation $kx^2 + (1-k)x + k = 0$ has equal real roots.
Find the possible values of *k*.

(b) (i) **10 marks** **Att 3**

2 (b) (i) $f(x)$ has no real roots; (it has two complex roots).
$g(x)$ has two real equal roots. [or: $g(x)$ has one real root]
$h(x)$ has two distinct real roots.

Blunders (-3)

B1 Does not state nature of roots, or states incorrect nature of roots.
B2 Does not state number of roots (once only).

Note: One blunder only in each function

(b) (ii) Quadratic **5 marks** **Att 2**
Values of *k* **5 marks** **Att 2**

2 (b) (ii)

Equal roots $\Rightarrow b^2 - 4ac = 0$.

$\therefore \quad (1-k)^2 - 4k^2 = 0$.

$1 - 2k + k^2 - 4k^2 = 0 \Rightarrow 3k^2 + 2k - 1 = 0$.

$(k+1)(3k-1) = 0 \Rightarrow k = -1,\ k = \frac{1}{3}$.

Blunders (-3)

B1 Indices
B2 Real equal roots condition
B3 Factors once only
B4 Roots formula once only
B5 Deduction of value from factor or no value from factor

Part (c) **20 (5, 5, 5, 5) marks** **Att (2, 2, 2, 2)**

(c)	**(i)**	One of the roots of $px^2 + qx + r = 0$ is n times the other root. Express r in terms of p, q and n.
	(ii)	One of the roots of $x^2 + qx + r = 0$ is five times the other. If q and r are positive integers, determine the set of possible values of q.

(c) (i) Root **5 marks** **Att 2**
Express r **5 marks** **Att 2**

2 (c) (i)

Roots are α and $n\alpha$.

$\therefore\ \alpha + n\alpha = \dfrac{-q}{p}$ and $\alpha(n\alpha) = \dfrac{r}{p}$.

$\alpha(1+n) = \dfrac{-q}{p} \Rightarrow \alpha = \dfrac{-q}{p(1+n)}$.

But $\alpha^2 = \dfrac{r}{pn} \Rightarrow \dfrac{q^2}{p^2(1+n)^2} = \dfrac{r}{pn}$.

$\therefore\ r = \dfrac{nq^2}{p(n+1)^2}$.

(c) (ii) r in terms of q **5 marks** **Att 2**
Values of q **5 marks** **Att 2**

2 (c) (ii)

$r = \dfrac{nq^2}{p(n+1)^2}$, by part **(i)**, where $n = 5$ and $p = 1$.

$\therefore r = \dfrac{5q^2}{36}$.

For r to be a positive integer, q^2 must be divisible by 36, so q is divisible by 6.

$\therefore\ q = \{6, 12, 18, 24, ...\}$.

OR

2 (c) (ii) Equation : $x^2 - (-q)x + (r) = 0$

Roots : $\alpha, 5\alpha$

$$x^2 - (\alpha + 5\alpha)x + (5\alpha^2) = 0$$

Equating Coefficients: (i) : $6\alpha = -q \Rightarrow \alpha = -\dfrac{q}{6}$

(ii) $5\alpha^2 = r$

$$5\left(-\frac{q}{6}\right)^2 = r$$

$$r = \frac{5q^2}{36}$$

For r to be a positive integer, q^2 must be divisible by 36, so q is divisible by 6.

$\therefore\ q = \{6, 12, 18, 24, ...\}$.

Blunders (-3)

B1 Indices
B2 Statement quadratic equation once only
B3 Incorrect sum roots
B4 Incorrect product roots
B5 One value of q only or two values q

Slips (-1)

S1 Numerical

QUESTION 3

Part (a)	**10 (5, 5) marks**	**Att (2, 2)**
Part (b)	**20 (5, 5, 5, 5) marks**	**Att (2, 2, 2, 2)**
Part (c)	**20 (5, 5, 5, 5) marks**	**Att (2, 2, 2, 2)**

Part (a)	**10 (5, 5) marks**	**Att (2, 2)**

3 (a) $z_1 = a + bi$ and $z_2 = c + di$, where $i^2 = -1$.

Show that $\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$, where $\bar{z}$ is the complex conjugate of z.

$\overline{z_1} + \overline{z_2}$	**5 marks**	**Att 2**
$\overline{z_1 + z_2}$	**5 marks**	**Att 2**

3 (a) $\overline{z_1} = a - bi,\ \overline{z_2} = c - di \Rightarrow \overline{z_1} + \overline{z_2} = (a+c) - (b+d)i.$

$\overline{z_1 + z_2} = \overline{(a+c) + (b+d)i} = (a+c) - (b+d)i = \overline{z_1} + \overline{z_2}.$

Blunders (-3)

B1 *i*

B2 Conjugate

Part (b) **20 (5, 5, 5, 5) marks** **Att (2, 2, 2, 2)**

(b) Let $A = \frac{1}{2}\begin{pmatrix} 1 & -\sqrt{3} \\ \sqrt{3} & 1 \end{pmatrix}$.

(i) Express A^3 in the form $\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$, where $a, b \in \mathbf{Z}$.

(ii) Hence, or otherwise, find A^{17}.

(b) (i) A^2	**5 marks**	**Att 2**
A^3	**5 marks**	**Att 2**

3 (b) (i)

$$A^2 = \frac{1}{4}\begin{pmatrix} 1 & -\sqrt{3} \\ \sqrt{3} & 1 \end{pmatrix}\begin{pmatrix} 1 & -\sqrt{3} \\ \sqrt{3} & 1 \end{pmatrix} = \frac{1}{4}\begin{pmatrix} -2 & -2\sqrt{3} \\ 2\sqrt{3} & -2 \end{pmatrix}.$$

$$\therefore A^3 = \frac{1}{8}\begin{pmatrix} 1 & -\sqrt{3} \\ \sqrt{3} & 1 \end{pmatrix}\begin{pmatrix} -2 & -2\sqrt{3} \\ 2\sqrt{3} & -2 \end{pmatrix} = \frac{1}{8}\begin{pmatrix} -8 & 0 \\ 0 & -8 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}.$$

(b) (ii) Values in A^{17}	**5 marks**	**Att 2**
A^{17} calculated	**5 marks**	**Att 2**

3 (b) (ii)

$$A^{17} = \left(A^3\right)^5 A^2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}^5 \frac{1}{4}\begin{pmatrix} -2 & -2\sqrt{3} \\ 2\sqrt{3} & -2 \end{pmatrix}$$

$$= \frac{1}{4}\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} -2 & -2\sqrt{3} \\ 2\sqrt{3} & -2 \end{pmatrix} = \frac{1}{4}\begin{pmatrix} 2 & 2\sqrt{3} \\ -2\sqrt{3} & 2 \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} 1 & \sqrt{3} \\ -\sqrt{3} & 1 \end{pmatrix}.$$

Blunders (-3)
B1 Indices

Slips (-1)
S1 Numerical
S2 Each incorrect element

Note: Can only get Att 2 in (ii) if A^3 not a diagonal matrix (in second 5 marks).

Part (c) **20 (5, 5, 5, 5) marks** **Att (2, 2, 2, 2)**

(c)	**(i)**	Use De Moivre's theorem to prove that $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$.
	(ii)	Hence, find $\int \sin^3\theta\, d\theta$.

(c) (i) Sin 3θ **5 marks** **Att 2**
Value **5 marks** **Att 2**

3 (c) (i)

$$(\cos\theta + i\sin\theta)^3 = \cos 3\theta + i\sin 3\theta.$$

$$(\cos\theta + i\sin\theta)^3 = \cos^3\theta + 3\cos^2\theta(i\sin\theta) + 3\cos\theta(i\sin\theta)^2 + (i\sin\theta)^3.$$

$$= \cos^3\theta - 3\cos\theta\sin^2\theta + 3i\cos^2\theta\sin\theta - i\sin^3\theta.$$

$$\therefore\ \sin 3\theta = 3\cos^2\theta\sin\theta - \sin^3\theta = 3\sin\theta\left(1-\sin^2\theta\right) - \sin^3\theta$$

$$= 3\sin\theta - 4\sin^3\theta.$$

(c) (ii) $\int \sin^3\theta . d\theta$ **5 marks** **Att 2**
Finish **5 marks** **Att 2**

3 (c) (ii) $\sin 3\theta = 3\sin\theta - 4\sin^3\theta \Rightarrow \sin^3\theta = \frac{1}{4}[3\sin\theta - \sin 3\theta].$

$$\therefore \int \sin^3\theta\, d\theta = \frac{1}{4}\int (3\sin\theta - \sin 3\theta)d\theta = \frac{1}{4}\left[-3\cos\theta + \frac{1}{3}\cos 3\theta\right] + C.$$

Note: Not "hence" $\Rightarrow$ zero marks for integration.

Blunders (-3)

B1 Statement De Moivre once only
B2 Binomial expansion once only
B3 *i*
B4 Indices
B5 Trig formula
B6 Not like to like when equating coefficients
B7 Integration
B8 *C* omitted

QUESTION 4

Part (a)	**10 (5, 5) marks**	**Att (2, 2)**
Part (b)	**20 (5, 5, 5, 5) marks**	**Att (2, 2, 2, 2)**
Part (c)	**20 (5, 5, 5, 5) marks**	**Att (2, 2, 2, 2)**

Part (a) **10 (5, 5) marks** **Att (2, 2)**

4. **(a)** Three consecutive terms of an arithmetic series are $4x+11$, $2x+11$, and $3x+17$. Find the value of *x*.

Definition of A.P.	**5 marks**	**Att 2**
Value *x*	**5 marks**	**Att 2**

4 (a)

$$(2x+11)-(4x+11)=(3x+17)-(2x+11).$$

$$-2x = x+6 \Rightarrow x=-2.$$

(And the three terms are $3,\ 7$ and 11.)

Blunders (-3)

B1 AP statement

Slips (-1)

S1 Numerical

Worthless

W1 Geometric sequence

W2 Puts in values for *x*

Part (b) **20 (5, 5, 5, 5) marks** **Att (2, 2, 2, 2)**

(b) **(i)** Show that $\dfrac{2}{r^2-1}=\dfrac{1}{r-1}-\dfrac{1}{r+1}$, where $r \neq \pm 1$.

(ii) Hence, find $\displaystyle\sum_{r=2}^{n}\frac{2}{r^2-1}$.

(iii) Hence, evaluate $\displaystyle\sum_{r=2}^{\infty}\frac{2}{r^2-1}$.

(b) (i) **5 marks** **Att 2**

4 (b) (i)

$$\frac{1}{r-1}-\frac{1}{r+1}=\frac{r+1-r+1}{(r-1)(r+1)}=\frac{2}{r^2-1}.$$

OR

4 (b) (i)

Let $\frac{2}{r^2-1}=\frac{a}{r-1}-\frac{b}{r+1}$

$$2=q(r+1)-b(r-1)$$

$$(0)r+(2)=(a-b)r+(a+b)$$

Equating Coefficients : (i) : $a-b=0$
(ii) : $a+b=2$

(i) : $a-b=0$
(ii) : $a+b=2$

$2a \quad =2$
$a \quad =1$

(i) $a-b=0 \Rightarrow a=b \Rightarrow a=b=1$

$$\frac{2}{r^2-1}=\frac{1}{r-1}-\frac{1}{r+1}$$

(b)(ii) Set up cancellation	**5 marks**	**Att 2**
Finish	**5 marks**	**Att 2**

4 (b) (ii)

$$\sum_{r=2}^{n}\frac{2}{r^2-1}=\sum_{r=2}^{n}\left(\frac{1}{r-1}-\frac{1}{r+1}\right)$$

$$=\sum_{r=2}^{n}\left(\frac{1}{r-1}\right)-\sum_{r=2}^{n}\left(\frac{1}{r+1}\right)$$

$$=\sum_{r=1}^{n-1}\frac{1}{r}-\sum_{r=3}^{n+1}\frac{1}{r}$$

$$=\left(1+\frac{1}{2}+\sum_{r=3}^{n-1}\frac{1}{r}\right)-\left(\sum_{r=3}^{n-1}\frac{1}{r}+\frac{1}{n}+\frac{1}{n+1}\right)$$

$$=\frac{3}{2}-\frac{1}{n}-\frac{1}{n+1}$$

OR

(b)(ii) Terms U_2 to U_n	5 marks	Att 2
Sum to n terms	5 marks	Att 2

4 (b) (ii)

$$U_n = \frac{1}{n^2-1} = \frac{1}{n-1} - \frac{1}{n+1}$$

$$U_{n-1} = \frac{1}{n-2} - \frac{1}{n}$$

$$U_{n-2} = \frac{1}{n-3} - \frac{1}{n-1}$$

$$\vdots \qquad \vdots$$

$$U_4 = \frac{1}{3} - \frac{1}{5}$$

$$U_3 = \frac{1}{2} - \frac{1}{4}$$

$$U_2 = \frac{1}{1} - \frac{1}{3}$$

$$S_n = 1 + \frac{1}{2} - \frac{1}{n} - \frac{1}{n+1}$$

$$S_n = \frac{3}{2} - \frac{1}{n} - \frac{1}{n+1}$$

(b) (iii) Sum to infinity	5 marks	Att 2

4 (b) (iii)

$$\sum_{r=2}^{\infty} \frac{2}{r^2-1} = \underset{n\to\infty}{\text{Limit}} \left(\frac{3}{2} - \frac{1}{n} - \frac{1}{n+1} \right) = \frac{3}{2}.$$

Blunders (-3)

B1 Indices
B2 Cancellation must be shown or implied
B3 Not like to like when equating coefficients
B4 Term omitted
B5 Gets S_r

Slips (-1)

S1 Numerical

Note: Must show three terms at start and two terms at finish or *vice versa*.

Part (c)	**20 (5, 5, 5, 5) marks**	**Att (2, 2, 2, 2)**

(c) A finite geometric sequence has first term *a* and common ratio *r*.
The sequence has $2m+1$ terms, where $m \in \mathbf{N}$.

(i) Write down the last term, in terms of *a*, *r*, and *m*.

(ii) Write down the middle term, in terms of *a*, *r*, and *m*.

(iii) Show that the product of all of the terms of the sequence is equal to the middle term raised to the power of the number of terms.

Part (c) (i)	**5 marks**	**Att 2**

4 (c) (i) Last term $= ar^{2m}$.

Part (c) (ii)	**5 marks**	**Att 2**

4 (c) (ii) Middle term $= ar^{m}$.

(c) (iii) Product	**5 marks**	**Att 2**
Show	**5 marks**	**Att 2**

4 (c) (iii)

Product of terms $= a \times ar \times ar^2 \times \ldots\ldots\ldots \times ar^{2m}$

$= a^{2m+1} \times r^{0+1+2+\ldots\ldots+2m}$. $\quad [0+1+2+\ldots\ldots+2m$ is an A.P. with $2m+1$ terms $]$

$= a^{2m+1}\left(r^{\frac{(2m+1)}{2}(2m)}\right) = a^{2m+1}r^{m(2m+1)}$

$= \left(ar^m\right)^{2m+1}$.

Blunders (-3)
B1 Indices
B2 $U_n \neq AR^{n-1}$
B3 Formula *AP*
B4 Incorrect substitution into formula once only
B5 Middle term

Slips (-1)
S1 Numerical

QUESTION 5

Part (a)	**10 (5, 5) marks**	**Att (2, 2)**
Part (b)	**20 (5, 5, 10) marks**	**Att (2, 2, 3)**
Part (c)	**20 (5, 5, 5, 5) marks**	**Att (2, 2, 2, 2)**

Part (a)	**10 (5, 5)marks**	**Att (2, 2)**

5 (a) **(a)** Solve for *x*: $x-2=\sqrt{3x-2}$.

Quadratic	**5 marks**	**Att 2**
Solution	**5 marks**	**Att 2**

5 (a)

$$x-2=\sqrt{3x-2} \Rightarrow (x-2)^2=3x-2.$$
$$x^2-4x+4=3x-2 \Rightarrow x^2-7x+6=0.$$
$$(x-6)(x-1)=0 \Rightarrow x=6 \text{ and } x=1.$$

Test: $x = 1$ LHS: $(x-2)=(1-2)=-1$

RHS: $\sqrt{3x-2}=\sqrt{1}=1$

$x \neq 1$

$x = 6$ LHS: $x-2=6-2=4$

RHS: $\sqrt{3x-2}=\sqrt{16}=4$

Solution: $x=6$

Blunders (-3)

B1 Indices
B2 Expansion $(x-2)^2$ once only
B3 Factors once only
B4 Roots formula once only
B5 Deduction value from factor
B6 Excess value

Slips (-1)

S1 Numerical

Attempts

A1 *x* = 6 and no other work merits Att 2
A2 *x* = 6 by trial and error merits Att 2

Part (b)	20 (5, 5, 10) marks	Att (2, 2, 3)

(b) Prove by induction that, for all positive integers n, 5 is a factor of $n^5 - n$.

P(1)	5 marks	Att 2
P(*k*)	5 marks	Att 2
P(*k*+1)	10 marks	Att 3

5 (b)

Let $P(n)$ be the proposition that 5 is a factor of $n^5 - n$.

Test $P(1)$: $1-1=0$, which is divisible by 5.

Assume $P(k)$: $k^5 - k$ is divisible by 5.

Try to deduce $P(k+1)$: that $(k+1)^5 - (k+1)$ is divisible by 5.

$$(k+1)^5 - (k+1) = k^5 + 5k^4 + 10k^3 + 10k^2 + 5k + 1 - k - 1.$$

$$= (k^5 - k) + 5(k^4 + 2k^3 + 2k^2 + k)$$

div. by 5 from P(k) has 5 as factor

so sum is divisible by 5, given $P(k)$.

We have $P(1)$ and $\{P(k) \Rightarrow P(k+1)\}$. Hence, $P(n)$ for all positive integers n.

OR

5 (b)

To prove : $(n^5 - n)$ is divisible by 5

$n = 1$: $1^5 - 1 = 0$, which is divisible by 5

$\Rightarrow$ true for $n = 1$

Assume true for $n = k$: $k^5 - k$ is divisible by 5.

To prove: $(k+1)^5 - (k+1)$ is divisible by 5.

Let $f(k) = k^5 - k$. Given the assumption that $f(k)$ is divisible by 5, then $f(k+1)$ will be divisible by 5 if and only if $[f(k+1) - f(k)]$ is divisible by 5.

Now, $f(k+1) - f(k) = [(k+1)^5 - (k+1)] - [k^5 - k]$

$= [k^5 + 5k^4 + 10k^3 + 10k^2 + 5k + 1 - k - 1] - k^5 + k$

$= 5k^4 + 10k^3 + 10k^2 + 5k$

$= 5(k^4 + 2k^3 + 2k^2 + k)$, which is divisible by 5.

So, the statement is true for $n = k + 1$ whenever it is true for $n = k$.

Since it is true for $n = 1$, then, by induction, it is true for all positive integers.

Blunders (-3)

B1 Binomial expansion once only

B2 Indices

B3 Expansion of $(k+1)^5$ once only

Note: Must prove $P(1)$ step (not sufficient to state $P(n)$ true for $n = 1$).

Part (c)	20 (5, 5, 5, 5) marks	Att (2, 2, 2, 2)

(c) Solve the simultaneous equations

$$\log_3 x + \log_3 y = 2$$
$$\log_3(2y-3) - 2\log_9 x = 1 \ .$$

One var. in terms of the other	**5 marks**	**Att 2**
Change of base	**5 marks**	**Att 2**
Quadratic	**5 marks**	**Att 2**
Solution	**5 marks**	**Att 2**

5 (c)

$$\log_3 x + \log_3 y = 2$$
$$\log_3(xy) = 2$$
$$xy = 9$$
$$x = \frac{9}{y}$$

$$\log_3(2y-3) - 2\log_9 x = 1$$
$$\log_3(2y-3) - 2\frac{\log_3 x}{\log_3 9} = 1$$
$$\log_3(2y-3) - \cancel{2}\frac{\log_3 x}{\cancel{2}} = 1$$
$$\log_3\left(\frac{2y-3}{x}\right) = 1$$
$$\frac{2y-3}{x} = 3$$

$$(2y-3)\frac{y}{9} = 3$$
$$2y^2 - 3y - 27 = 0$$
$$(2y-9)(y+3) = 0$$

$y > 0 \Rightarrow y \neq -3$, so $y = \frac{9}{2}$, giving $x = 2$.

Blunders (-3)
B1 Logs
B2 Indices
B3 Formula change of base
B4 Factors
B5 Roots formula
B6 Deduction root from factor or no deduction
B7 Excess value

Worthless
W1 Drops “logs”

Note Must have a quadratic equation for last 5 marks

QUESTION 6

Part (a)	**10 marks**	**Att 3**
Part (b)	**20 (5, 5, 5, 5) marks**	**Att (2, 2, 2, 2)**
Part (c)	**20 (5, 5, 5, 5) marks**	**Att (2, 2, 2, 2)**

Part (a) **10 marks** **Att 3**

6 (a) Differentiate $\sin(3x^2 - x)$ with respect to x.

6 (a)

$$f(x) = \sin(3x^2 - x) \;\Rightarrow\; f'(x) = \cos(3x^2 - x).(6x - 1).$$

Blunders (-3)
B1 Differentiation

Attempts
A1 Error in differentiation formula

Part (b) **15 (5, 5, 5) marks** **Att (2, 2, 2)**

(b) **(i)** Differentiate $\sqrt{x}$ with respect to x, from first principles.

(ii) An object moves in a straight line such that its distance from a fixed point is given by $s = \sqrt{t^2 + 1}$, where s is in metres and t is in seconds.
Find the speed of the object when $t = 5$ seconds.

(b)(i) $f(x+h) - f(x)$	**5 marks**	**Att 2**
Multiplication	**5 marks**	**Att 2**
Finish	**5 marks**	**Att 2**

6 (b) (i)

$$f(x) = \sqrt{x} \Rightarrow f(x) = \sqrt{x+h}$$

$$f(x+h) - f(x) = \sqrt{x+h} - \sqrt{x}$$

$$= \frac{\left(\sqrt{x+h} - \sqrt{x}\right)}{1} \times \frac{\left(\sqrt{x+h} + \sqrt{x}\right)}{\left(\sqrt{x+h} + \sqrt{x}\right)}$$

$$= \frac{x + h - x}{\sqrt{x+h} + \sqrt{x}}$$

$$= \frac{h}{\sqrt{x+h} + \sqrt{x}}$$

$$\therefore \underset{h \to 0}{\text{Limit}} \frac{f(x+h) - f(x)}{h} = \underset{h \to 0}{\text{Limit}} \frac{1}{\sqrt{x+h} + \sqrt{x}} = \frac{1}{2\sqrt{x}}$$

OR

6 (b) (i)

$$y = \sqrt{x}$$
$$y + \Delta y = \sqrt{x + \Delta x}$$
$$\Delta y = \sqrt{x + \Delta x} - \sqrt{x}$$
$$\frac{\Delta y}{\Delta x} = \frac{\sqrt{x + \Delta x} - \sqrt{x}}{\Delta x} \cdot \frac{\sqrt{x + \Delta x} + \sqrt{x}}{\sqrt{x + \Delta x} + \sqrt{x}}$$
$$= \frac{\left(\sqrt{x + \Delta x}\right)^2 - \left(\sqrt{x}\right)^2}{\Delta x\left[\sqrt{x + \Delta x} + \sqrt{x}\right]}$$
$$= \frac{x + \Delta x - x}{\Delta x\left(\sqrt{x + \Delta x} + \sqrt{x}\right)}$$
$$\frac{\Delta y}{\Delta x} = \frac{1}{\sqrt{x + \Delta x} + \sqrt{x}}$$
$$\underset{\Delta x \to 0}{\text{Limit}} \frac{\Delta y}{\Delta x} = \frac{1}{\sqrt{x} + \sqrt{x}} = \frac{1}{2\sqrt{x}}$$

Blunders (-3)
B1 $f(x+h)$ or $(x + \Delta x)$
B2 Indices
B3 No limits shown or implied or no indication $h \to 0$
B4 $h \to \infty$
B5 Conjugate
B6 No left hand side

Worthless
W1 Not 1st principles

(b) (ii) **5 marks** **Att 2**

6 (b) (ii)

$$s = \left(t^2 + 1\right)^{\frac{1}{2}} \quad \Rightarrow \quad \frac{ds}{dt} = \frac{1}{2}\left(t^2 + 1\right)^{-\frac{1}{2}}.2t = \frac{t}{\sqrt{t^2 + 1}}.$$
$$\therefore \text{ At } t = 5, \ \frac{ds}{dt} = \frac{5}{\sqrt{26}} \text{ metres per second.}$$

Blunders (-3)
B1 Differentiation
B2 Indices
B3 No substitution $t = 5$

Slips (-1)
S1 Incorrect units or omitted units

Attempts
A1 Error in differentiation formula

Part (c) **20 (5, 5, 5, 5) marks** **Att (2, 2, 2, 2)**

(c) The equation of a curve is $y = \frac{2}{x-3}$.

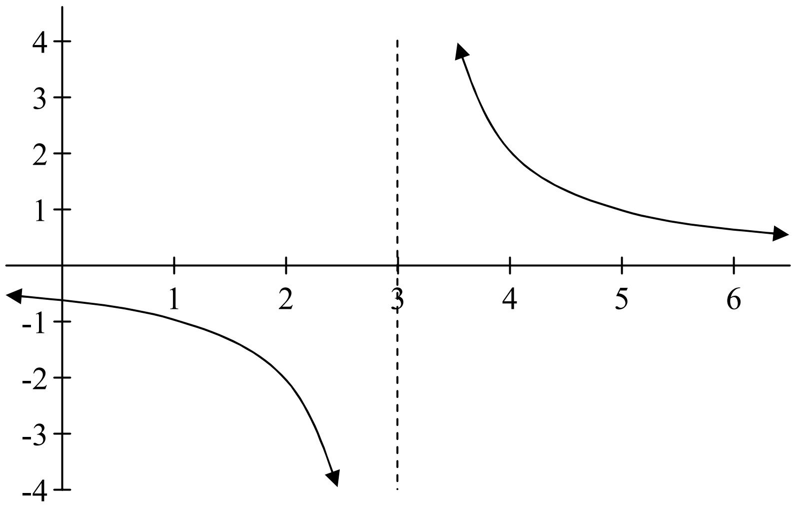
(i) Write down the equations of the asymptotes and hence sketch the curve.

(ii) Prove that no two tangents to the curve are perpendicular to each other.

(c) (i) Asymptotes **5 marks** **Att 2**
Sketch **5 marks** **Att 2**

6 (c) (i)

Equations of asymptotes are $x = 3$ and $y = 0$.



(c) (ii) Slope **5 marks** **Att 2**
Deduction **5 marks** **Att 2**

6 (c) (ii)

$$y = \frac{2}{x-3} = 2(x-3)^{-1} \quad \Rightarrow \quad \frac{dy}{dx} = -2(x-3)^{-2} = \frac{-2}{(x-3)^2}.$$

$\therefore$ Slope of tangent at (x, y) is $m = \frac{-2}{(x-3)^2}$.

But m will be negative for all values of $x \Rightarrow m_1.m_2 \neq -1$

$\therefore$ No two tangents are perpendicular to each other.

OR

6 (c) (ii)

$$y = 2(x-3)^{-1}$$

$$m = \frac{dy}{dx} = \frac{-2}{(x-3)^2}$$

Let tangents at $x = a$ and $x = b$ be perpendicular

At $x = a$: $m_1 = \frac{-2}{(a-3)^2}$

At $x = b$: $m_2 = \frac{-2}{(b-3)^2}$

$$(m_1).(m_2) = \frac{-2}{(a-3)^2} \cdot \frac{-2}{(b-3)^2} = \frac{4}{(a-3)^2.(b-3)^2} \neq -1, \text{ (since LHS is positive).}$$

$\Rightarrow$ Tangents cannot be perpendicular.

Blunders (-3)
B1 Indices
B2 Asymptote
B3 Differentiation
B4 Slope $\neq \frac{dy}{dx}$
B5 $m_1 m_2 \neq -1$
B6 Incorrect deduction or no deduction

Slips
S1 Curve not approaching asymptotes.

Attempts
A1 Error in differentiation formula

Worthless
W1 Integration

QUESTION 7

Part (a)	**10 (5, 5) marks**	**Att (2, 2)**
Part (b)	**20 (5, 5, 5, 5) marks**	**Att (2, 2, 2, 2)**
Part (c)	**20 (5, 5, 5, 5) marks**	**Att (2, 2, 2, 2)**

Part (a)	**10 (5, 5) marks**	**Att (2, 2)**

7 (a) **(a)** The equation of a curve is $x^2 - y^2 = 25$. Find $\frac{dy}{dx}$ in terms of x and y.

Differentiate	**5 marks**	**Att 2**
Isolate $\frac{dy}{dx}$	**5 marks**	**Att 2**

7 (a) $x^2 - y^2 = 25 \Rightarrow 2x - 2y\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = \frac{x}{y}$.

OR

7 (a)

$$x^2 - y^2 = 25$$
$$y^2 = x^2 - 25$$

$$y = \sqrt{x^2 - 25} \qquad \text{OR} \qquad y = -\sqrt{x^2 - 25}$$

$$y = \left(x^2 - 25\right)^{\frac{1}{2}} \qquad\qquad y = -\left(x^2 - 25\right)^{\frac{1}{2}}$$

$$\frac{dy}{dx} = \frac{1}{2}\left(x^2 - 25\right)^{-\frac{1}{2}}.2x \qquad\qquad \frac{dy}{dx} = -\left[\frac{1}{2}\left(x^2 - 25\right)^{-\frac{1}{2}}.2x\right]$$

$$= \frac{x}{\sqrt{x^2 - 25}} \qquad\qquad = -\left[\frac{x}{\sqrt{x^2 - 25}}\right]$$

$$= \frac{x}{y} \qquad\qquad = \frac{x}{y}$$

$$\frac{dy}{dx} = \frac{x}{y}$$

Blunders (-3)
B1 Differentiation
B2 Indices

Attempts
A1 Error in differentiation formula
A2 $\frac{dy}{dx} = 2x - 2y\frac{dy}{dx}$ and uses two $\frac{dy}{dx}$ terms in first 5 marks.

Worthless
W1 No differentiation
W2 Integration

Part (b)	**20 (5, 5, 5, 5) marks**	**Att (2, 2, 2, 2)**

(b) A curve is defined by the parametric equations

$$x = \frac{3t}{t^2 - 2} \text{ and } y = \frac{6}{t^2 - 2}, \text{ where } t \neq \pm\sqrt{2}.$$

(i) Find $\frac{dy}{dx}$ in terms of t.

(ii) Find the equation of the tangent to the curve at the point given by $t = 2$.

(b) (i) $\frac{dx}{dt}, \frac{dy}{dt}$	**5 marks**	**Att 2**
$\frac{dy}{dx}$	**5 marks**	**Att 2**

7 (b) (i)

$$x = \frac{3t}{t^2 - 2} \quad \Rightarrow \quad \frac{dy}{dt} = \frac{3(t^2 - 2) - 3t.2t}{(t^2 - 2)^2} = \frac{-3t^2 - 6}{(t^2 - 2)^2}.$$

$$y = \frac{6}{t^2 - 2} = 6(t^2 - 2)^{-1} \quad \Rightarrow \quad \frac{dy}{dt} = -6(t^2 - 2)^{-2}.2t = \frac{-12t}{(t^2 - 2)^2}.$$

$$\frac{dy}{dx} = \frac{dy}{dt}.\frac{dt}{dx} = \frac{-12t}{(t^2 - 2)^2}.\frac{(t^2 - 2)^2}{-3t^2 - 6} = \frac{12t}{3t^2 + 6} = \frac{4t}{t^2 + 2}.$$

(b)(ii) Slope, point	**5 marks**	**Att 2**
Equation	**5 marks**	**Att 2**

7 (b) (ii)

$$t = 2 \quad \Rightarrow \quad x = \frac{6}{2} = 3 \quad \text{and} \quad t = 2 \quad \Rightarrow \quad y = \frac{6}{2} = 3. \quad \therefore \text{ Point is } (3, 3).$$

Slope of tangent at $t = 2$ is $\frac{8}{6} = \frac{4}{3}$.

$\therefore$ Equation of tangent: $y - 3 = \frac{4}{3}(x - 3) \quad \Rightarrow \quad 4x - 3y - 3 = 0.$

Blunders (-3)

B1 Differentiation

B2 Indices

B3 Error in getting $\frac{dy}{dx}$

B4 Equation of tangent

B5 Error in slope formula.

Slips (-1)

S1 Numerical

Attempts

A1 Error in differentiation formula

Part (c)	**20 (5, 5, 5, 5)**	**Att (2, 2, 2, 2)**

(c) The function $f(x)=x^3-3x^2+3x-4$ has only one real root.

(i) Show that the root lies between 2 and 3.

Anne and Barry are each using the Newton-Raphson method to approximate the root. Anne is starting with 2 as a first approximation and Barry is starting with 3.

(ii) Show that Anne's starting approximation is closer to the root than Barry's. (That is, show that the root is less than 2·5.)

(iii) Show, however, that Barry's next approximation is closer to the root than Anne's.

(c) (i)	**5 marks**	**Att 2**

$f(x)=x^3-3x^2+3x-4$.

$f(2)=8-12+6-4=-2<0$.

$f(3)=27-27+9-4=5>0$.

$\therefore$ root lies between 2 and 3.

(c) (ii)	**5 marks**	**Att 2**

$f(2.5)=(2.5)^3-3(2.5)^2+3(2.5)-4$

$=15.625-18.75+7.5-4$

$=0.375$

$f(2)<0$ and $f(2.5)>0$. $\therefore$ root is between 2 and 2.5.

So, root is closer to 2 than to 3.

(c) (iii) Formula + Differentiation	**5 marks**	**Att 2**
Finish	**5 marks**	**Att 2**

7 (c) (iii)

$x_{n+1}=x_n-\dfrac{f(x_n)}{f'(x_n)}$, where $f(x)=x^3-3x^2+3x-4$ and $f'(x)=3x^2-6x+3$

Ann: $f(2)=-2$ and $f'(2)=3$. $x_2=2-\dfrac{f(2)}{f'(2)}=2-\dfrac{-2}{3}=2\dfrac{2}{3}=2.666...$

Barry: $f(3)=5$ and $f'(3)=12$. $x_2=3-\dfrac{f(3)}{f'3}=3-\dfrac{5}{12}=2\dfrac{7}{12}=2.583...$

Both of these are above the root, so the lower one is closer (i.e. Barry's).

Blunders (-3)

B1 Indices

B2 Incorrect deduction from *f*(2) and *f* (3) or no deduction

B3 No *f*(2.5)

B4 Newton –Raphson formula

B5 Differentiation

B6 Incorrect deduction or no deduction from work in (iii)

QUESTION 8

Part (a)	**10 marks**	**Att 3**
Part (b)	**20 (5, 5, 5, 5) marks**	**Att (2, 2, 2, 2)**
Part (c)	**20 (5, 5, 5, 5) marks**	**Att (2, 2, 2, 2)**

Part (a) **10 marks** **Att 3**

8 (a) Find $\int\left(6x+3+\frac{1}{x^2}\right)dx.$

8 (a)

$$\int\left(6x+3+\frac{1}{x^2}\right)dx = 3x^2+3x-\frac{1}{x}+C.$$

Blunders (-3)
B1 Integration
B2 Indices
B3 No *c*

Attempts
A1 Only *c* correct

Worthless
W1 Differentiation for integration

Part (b) **20 (5, 5, 5, 5) marks** **Att (2, 2, 2, 2)**

(b) Evaluate **(i)** $\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \sin 3x \sin x\, dx$ **(ii)** $\int_{\ln 3}^{\ln 8} e^x\sqrt{1+e^x}\, dx$.

Integration **5 marks** **Att 2**
Value **5 marks** **Att 2**

8 (b) (i)

$$\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \sin 3x \sin x\, dx = \frac{1}{2}\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}(\cos 2x-\cos 4x)dx = \frac{1}{2}\left[\frac{1}{2}\sin 2x-\frac{1}{4}\sin 4x\right]_{-\frac{\pi}{4}}^{\frac{\pi}{4}}$$

$$=\frac{1}{2}\left[\left(\frac{1}{2}\sin\frac{\pi}{2}-\frac{1}{4}\sin\pi\right)-\left(\frac{1}{2}\sin\left(\frac{-\pi}{2}\right)-\frac{1}{4}\sin(-\pi)\right)\right]$$

$$=\frac{1}{2}\left[\left(\frac{1}{2}-0\right)-\left(-\frac{1}{2}-0\right)\right]=\frac{1}{2}.$$

Integration	5 marks	Att 2
Value	5 marks	Att 2

8 (b) (ii)

Let $u = 1 + e^x$. $\therefore\ du = e^x dx$.

$$\int_{\ln 3}^{\ln 8} e^x \sqrt{1+e^x}\ dx = \int_{1+e^{\ln 3}}^{1+e^{\ln 8}} u^{\frac{1}{2}} du, \text{ but } e^{\ln 8} = 8 \text{ and } e^{\ln 3} = 3.$$

$$= \int_4^9 u^{\frac{1}{2}} du = \left[\frac{2}{3}u^{\frac{3}{2}}\right]_4^9 = \frac{2}{3}[27-8] = \frac{38}{3}.$$

OR

8 (b) (ii)

Using x limits:

$$\int_{\ln 3}^{\ln 8} e^x \sqrt{1+e^x}\ dx = \frac{2}{3}u^{\frac{3}{2}}\Big|_{x=\ln 3}^{x=\ln 8}$$

$$= \frac{2}{3}\left(1+e^x\right)^{\frac{3}{2}}\Big|_{\ln 3}^{\ln 8}$$

$$= \frac{2}{3}\left[\left(1+e^{\ln 8}\right)^{\frac{3}{2}} - \left(1+e^{\ln 3}\right)^{\frac{3}{2}}\right]$$

$$= \frac{2}{3}\left[(9)^{\frac{3}{2}} - (4)^{\frac{3}{2}}\right]$$

$$= \frac{2}{3}[27-8] = \frac{2}{3}(19) = \frac{38}{3}$$

OR

8 (b) (ii)

$$\int_{\ln 3}^{\ln 8} e^x \sqrt{1+e^x}\ dx \qquad \text{Let } u = e^x$$

$$= \int \sqrt{1+e^x}.e^x dx \qquad \frac{du}{dx} = e^x$$

$$= \int (1+u)^{\frac{1}{2}} du \qquad du = e^x dx$$

$$= \frac{2}{3}(1+u)^{\frac{3}{2}}\Big|_{x=\ln 3}^{x=\ln 8}$$

$$= \frac{2}{3}\left(1+e^x\right)^{\frac{3}{2}}\Big|_{\ln 3}^{\ln 8}$$

$$= \frac{2}{3}\left[\left(1+e^{\ln 8}\right)^{\frac{3}{2}} - \left(1+e^{\ln 3}\right)^{\frac{3}{2}}\right]$$

$$= \frac{2}{3}\left[(1+8)^{\frac{3}{2}} - (1+3)^{\frac{3}{2}}\right]$$

$$= \frac{2}{3}\left[(9)^{\frac{3}{2}} - (4)^{\frac{3}{2}}\right] \qquad = \frac{2}{3}[27-8] = \frac{2}{3}(19) = \frac{38}{3}$$

* Incorrect substitution and unable to finish yields attempt at most

Blunders (-3)

B1 Trig formula
B2 Integration
B3 Differentiation
B4 Limits
B5 Incorrect order in applying limits
B6 Not calculating substituted limits
B7 Not changing limits
B8 Indices
B9 Logs
B10 $e^{\ln a} \neq a$

Slips (-1)

S1 Numerical
S2 Trig value
S3 Answer not tidied up

Worthless

W1 Differentiation instead of integration except where other work merits attempts

Part (c)	**20 (5, 5, 5, 5) marks**	**Att (2, 2, 2, 2)**

(c) Use integration methods to establish the standard formula for the volume of a cone.

Diagram + slope	**5 marks**	**Att 2**
Correct subst. into volume formula	**5 marks**	**Att 2**
Integration	**5 marks**	**Att 2**
Volume	**5 marks**	**Att 2**

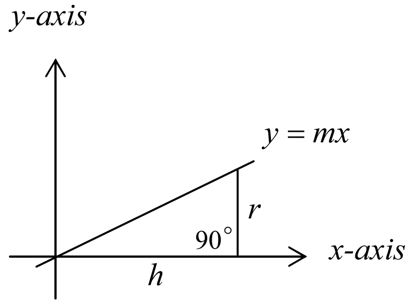
8 (c)

$$y = mx \quad \Rightarrow \quad y = \frac{r}{h}x.$$

$$\text{Volume of cone} = \pi \int_0^h y^2\,dx, \text{ where } y = \frac{r}{h}x.$$

$$V = \pi \int_0^h \frac{r^2}{h^2}x^2\,dx = \frac{1}{3}\pi\frac{r^2}{h^2}\left[x^3\right]_0^h = \frac{1}{3}\pi\frac{r^2}{h^2}h^3$$

$$V = \frac{1}{3}\pi r^2 h.$$



Blunders (-3)
B1 Integration
B2 Slope of line
B3 Equation of line
B4 Volume formula provided it is quadratic
B5 Limits
B6 No Limits
B7 Incorrect order in applying limits
B8 Indices

Slips (-1)
S1 Numerical

Attempts
A1 Uses $v = \pi y$

Worthless
W1 Differentiation instead of integration

Coimisiún na Scrúduithe Stáit
State Examinations Commission

LEAVING CERTIFICATE 2009

MARKING SCHEME

MATHEMATICS - PAPER 2

HIGHER LEVEL

GENERAL GUIDELINES FOR EXAMINERS – PAPER 2

1. Penalties of three types are applied to candidates' work as follows:
   - Blunders - mathematical errors/omissions (-3)
   - Slips - numerical errors (-1)
   - Misreadings (provided task is not oversimplified) (-1).

   Frequently occurring errors to which these penalties must be applied are listed in the scheme. They are labelled: B1, B2, B3,…, S1, S2,…, M1, M2,…etc. These lists are not exhaustive.

2. When awarding attempt marks, e.g. Att(3), note that
   - any *correct, relevant* step in a part of a question merits at least the attempt mark for that part
   - if deductions result in a mark which is lower than the attempt mark, then the attempt mark must be awarded
   - a mark between zero and the attempt mark is never awarded.

3. Worthless work is awarded zero marks. Some examples of such work are listed in the scheme and they are labelled as W1, W2,…etc.

4. The phrase "hit or miss" means that partial marks are not awarded – the candidate receives all of the relevant marks or none.

5. The phrase "and stops" means that no more work of merit is shown by the candidate.

6. Special notes relating to the marking of a particular part of a question are indicated by an asterisk. These notes immediately follow the box containing the relevant solution.

7. The sample solutions for each question are not intended to be exhaustive lists – there may be other correct solutions. Any examiner unsure of the validity of the approach adopted by a particular candidate to a particular question should contact his/her advising examiner.

8. Unless otherwise indicated in the scheme, accept the best of two or more attempts – even when attempts have been cancelled.

9. The *same* error in the *same* section of a question is penalised *once* only.

10. Particular cases, verifications and answers derived from diagrams (unless requested) qualify for attempt marks at most.

11. A serious blunder, omission or misreading results in the attempt mark at most.

12. Do not penalise the use of a comma for a decimal point, e.g. €5.50 may be written as €5,50.

QUESTION 1

Part (a)	**10 (5, 5) marks**	**Att (2, 2)**
Part (b)	**20 (10, 10) marks**	**Att (3, 3)**
Part (c)	**20 (10, 5, 5) marks**	**Att (3, 2, 2)**

Part (a) **10(5, 5) marks** **Att (2, 2)**

1 (a) Show that, for all values of $t \in \mathbf{R}$, the point $\left(\frac{2t}{1+t^2}, \frac{1-t^2}{1+t^2}\right)$ lies on the circle $x^2 + y^2 = 1$.

Part (a) Substitution **5 marks** **Att 2**
Finish **5 marks** **Att 2**

1 (a)

$$x^2 + y^2 = \frac{4t^2}{(1+t^2)^2} + \frac{(1-t^2)^2}{(1+t^2)^2} = \frac{4t^2 + 1 - 2t^2 + t^4}{(1+t^2)^2} = \frac{1+2t^2+t^4}{(1+t^2)^2} = \frac{(1+t^2)^2}{(1+t^2)^2} = 1.$$

Blunders (-3)
B1 Incorrect squaring (apply once if same type of error)
B2 Incorrect factors
B3 Incorrect conclusion

Slips (-1)
S1 Arithmetic error

Attempts (2, 2 marks)
A1 Some correct substitution for x or y
A2 Effort at expressing t^2 in terms of y

Part (b) **20 (10, 10) marks** **Att (3, 3)**

(b) **(i)** Find the equation of the tangent to the circle $x^2 + y^2 = 10$ at the point $(3, 1)$.

(ii) Find the values of $k \in \mathbf{R}$ for which the line $x - y + k = 0$ is a tangent to the circle $(x-3)^2 + (y+4)^2 = 50$.

Part (b) (i) **10 marks** **Att 3**

1 (b) (i)

Equation of tangent: $xx_1 + yy_1 = r^2 \Rightarrow 3x + y = 10.$

or

Centre of circle (0,0) $\Rightarrow$ Slope diameter = $\frac{1}{3}$ $\Rightarrow$ Slope Tangent = -3

Equation of tangent: $y-1 = -3(x-3)$

Blunders (-3)
B1 Error in slope formula
B2 Slope of tangent not perpendicular to the diameter
B3 Error in equation of line formula
B4 Error in equation of tangent formula
B5 Incorrect centre of circle

Slips (-1)
S1 Arithmetic error

Attempts (3 marks)
A1 Equation of tangent formula
A2 Slope of diameter only
A3 Equation of line with some substitution

Part (b) (ii) **10 marks** **Att 3**

1 (b) (ii) Centre $(3,-4)$ and radius $= \sqrt{50} = 5\sqrt{2}$.
Since a tangent, perpendicular distance from centre $(3,-4)$ to $x - y + k = 0$ equals radius.

$$\therefore \left|\frac{3+4+k}{\sqrt{2}}\right| = 5\sqrt{2} \;\Rightarrow\; |7+k| = 10 \Rightarrow 7+k = \pm 10. \;\therefore\; k = 3 \text{ or } k = -17.$$

OR

Part (b) (ii) **10 marks** **Att 3**

$y = x + k$

$(x-3)^2 + ((x+k)+4)^2 = 50$

$2x^2 + (2+2k)x + (8k+25) = 0$

One point of contact $\Rightarrow (2+2k)^2 - 4.2(k^2 + 8k - 25) = 0$

$\Rightarrow k^2 + 14k - 51 = 0$

$\Rightarrow (k-3)(k+17) = 0$

$\Rightarrow k = 3, k = -17$

Blunders (-3)
B1 Incorrect centre of circle
B2 Error in perpendicular distance formula
B3 Incorrect radius
B4 One value of *k* only
B5 Incorrect squaring
B6 Error in factors

Slips (-1)
S1 Arithmetic error

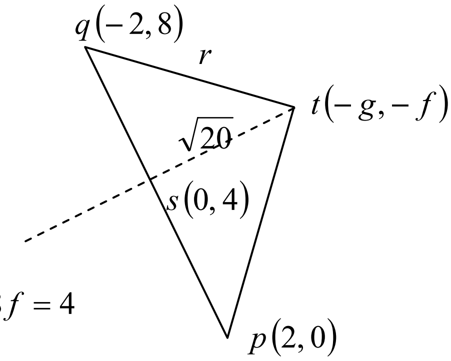
Attempts (3 marks)
A1 Centre or radius correct
A2 Some correct substitution into perpendicular formula
A3 Some correct substitution of $y = x + k$ or equivalent into circle

Part (c) **20 (10, 5, 5) marks** **Att (3, 2, 2)**

(c) Two circles intersect at $p(2,0)$ and $q(-2,8)$. The distance from the centre of each circle to the common chord $[pq]$ is $\sqrt{20}$.
Find the equations of the two circles.

Part (c) First equation in *f* and *g*	**10 marks**	**Att 3**
Equation in one variable	**5 marks**	**Att 2**
Finish	**5 marks**	**Att 2**

1 (c)



Slope $pq = \dfrac{8-0}{-2-2} = -2 \Rightarrow$ slope $st = \dfrac{1}{2}$.

$\therefore \dfrac{4+f}{0+g} = \dfrac{1}{2} \Rightarrow g = 2f + 8.$

$|st|^2 = 20 \Rightarrow (0+g)^2 + (4+f)^2 = 20 \;\; \Rightarrow \;\; g^2 + f^2 + 8f = 4$

$\Rightarrow (2f+8)^2 + f^2 + 8f = 4 \;\; \Rightarrow \;\; 5f^2 + 40f + 60 = 0.$

$\therefore f^2 + 8f + 12 = 0 \;\; \Rightarrow \;\; (f+2)(f+6) = 0.$

$f = -2 \Rightarrow g = 4$ or $f = -6 \Rightarrow g - 4.$

$\therefore$ Centres are $(-4,2)$ and $(4,6)$, $\quad r = \sqrt{40}$.

Circles are: $(x+4)^2 + (y-2)^2 = 40$ and $(x-4)^2 + (y-6)^2 = 40.$

or $x^2 + y^2 + 8x - 4y - 20 = 0$ and $x^2 + y^2 - 8x - 12y + 12 = 0$

OR

Part (c) First equation in *f* and *g*	**10 marks**	**Att 3**
Equation in one variable	**5 marks**	**Att 2**
Finish	**5 marks**	**Att 2**

Slope $pq \; = \dfrac{0-8}{2--2} = -2$

Equation pq: $\quad y = -2(x-2)$ or $2x + y - 4 = 0$

Perp. distance $(-g, -f)$ to pq: $\left|\dfrac{-2g-f-4}{\sqrt{5}}\right| = \sqrt{20}$

$\Rightarrow -2g - f - 4 = \pm 10 \Rightarrow 2g + f + 14 = 0$ and $2g + f - 6 = 0$

Distance from (0,4) (= midpoint pq) to $(-g, -f) \Rightarrow (0+g)^2 + (4+f)^2 = 20$

Solving between $g^2 + (4+f)^2 = 20$ and $2g + f - 6 = 0$ gives $\quad g = 4$ and $f = -2$

Solving between $g^2 + (4+f)^2 = 20$ and $2g + f + 14 = 0$ gives $\quad g = -4$ and $f = -6$

Eq. 1: $x^2 + y^2 + 8x - 4y + c = 0$

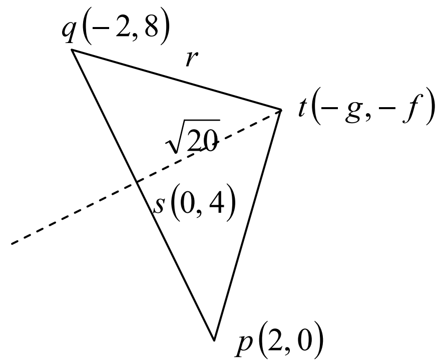
(2, 0) on circle $\Rightarrow c = -20 \Rightarrow x^2 + y^2 + 8x - 4y - 20 = 0$

Eq2: Same method $\Rightarrow c = 12 \Rightarrow x^2 + y^2 - 8x - 12y + 12 = 0$

OR

Part (c)	**First equation in *f* and *g***	**10 marks**	**Att 3**
	Equation in one variable	**5 marks**	**Att 2**
	Finish	**5 marks**	**Att 2**

1 (c)



$(2,0) \in \text{Circle} \Rightarrow 2^2 + 0 + 2g(2) + 2f(o) + c = 0$

$\Rightarrow 4g + c = -4 \Rightarrow c = -4g - 4$

$(-2,8) \in \text{Circle} \Rightarrow -4g + 16f + c + 68 = 0$

$\Rightarrow -4g + 16f - 4g - 4 + 68 = 0 \Rightarrow g = 2(f + 4)$

$s(\text{midpoint}) = (0,4)$

But $\sqrt{g^2 + (4+f)^2} = \sqrt{20} \Rightarrow g^2 + (4+f)^2 = 20$

$\Rightarrow (2(f+4))^2 + (4+f)^2 = 20 \Rightarrow 5(f+4)^2 = 20$

$\Rightarrow (f+4)^2 = 4 \Rightarrow f + 4 = \pm 2 \Rightarrow f = -6 \text{ and } -2$

$f = -6 \Rightarrow g = -4 \Rightarrow c = 12$

$f = -2 \Rightarrow g = 4 \Rightarrow c = -20$

Circles:

$x^2 + y^2 + 8x - 4y - 20 = 0$

$x^2 + y^2 - 8x - 12y + 12 = 0$

OR

Part (c)	**First equation in *f* and g**	**10marks**	**Att 3**
	Equation in one variable	**5 marks**	**Att 2**
	Finish	**5 marks**	**Att 2**

$|pq| = \sqrt{(2+2)^2 + (0-8)^2} = \sqrt{80} \Rightarrow |ps| = \sqrt{20}$

$|pt|^2 = 20 + 20 = 40 \Rightarrow |pt| = \sqrt{40}$

$\therefore p(2,0)$ as centre of a circle with radius $\sqrt{40}$

$\Rightarrow (x-2)^2 + y^2 = 40$

But $(-g, -f)$ on circle

$\Rightarrow (-g-2)^2 + (0+f)^2 = 40$

st is a chord.

Slope $pq = \dfrac{8-2}{-2-2} = -2 \Rightarrow$ slope $st = \dfrac{1}{2}$

$\Rightarrow \dfrac{4+f}{0+g} = \dfrac{1}{2} \Rightarrow g = 2f + 8$

$\therefore (-2f - 8 - 2)^2 + f^2 = 40$

$\Rightarrow 5f^2 + 40f + 60 = 0 \Rightarrow f^2 + 8f + 12 = 0$

$\Rightarrow (f+6)(f+2) = 0 \Rightarrow f = -2$ and $f = -6$

$f = -6 \Rightarrow g = -4 \Rightarrow c = 12$

$f = -2 \Rightarrow g = 4 \Rightarrow c = -20$

Circles

$x^2 + y^2 + 8x - 4y - 20 = 0$

$x^2 + y^2 - 8x - 12y + 12 = 0$

Blunders (-3)

B1 Error in distance formula
B2 Error in mid point formula
B3 Error in perpendicular distance formula
B4 Incorrect application of Pythagoras formula
B5 Error in slope formula
B6 Error in squaring
B7 Error in factors
B8 Equation of one circle only

Slips (-1)

S1 Arithmetic error

Attempts (3,2,2 marks)

A1 Mid point or slope pq
A2 c expressed in terms of g
A3 Radius only

QUESTION 2

Part (a)	**10 (5, 5) marks**	**Att (2, 2)**
Part (b)	**20 (10, 10) marks**	**Att (3, 3)**
Part (c)	**20 (10, 10) marks**	**Att (3, 3)**

Part (a)	**10 (5, 5) marks**	**Att (2, 2)**

2 (a)

(a) If $\vec{a} = 2\vec{i} + \vec{j}$, $\vec{b} = -\vec{i} + 5\vec{j}$, find the unit vector in the direction of $\vec{ab}$.

Part (a) $\vec{ab}$**.**	**5 marks**	**Att 2**
Finish	**5 marks**	**Att 2**

2 (a)

$$\vec{ab} = \vec{b} - \vec{a} = -\vec{i} + 5\vec{j} - 2\vec{i} - \vec{j} = -3\vec{i} + 4\vec{j}.$$

$$\left|\vec{ab}\right| = \left|-3\vec{i} + 4\vec{j}\right| = \sqrt{9+16} = 5.$$

$$\text{Unit vector} = \frac{\vec{ab}}{\left|\vec{ab}\right|} = \frac{-3\vec{i} + 4\vec{j}}{5} = -\frac{3}{5}\vec{i} + \frac{4}{5}\vec{j}.$$

Blunders (-3)

B1 Error in $\vec{ab} = \vec{b} - \vec{a}$
B2 Error in formula for norm of vector
B3 Answer not expressed in correct form

Slips (-1)
S1 Arithmetic error

Attempts (2,2 marks)
A1 Norm formula with some substitution
A2 $\vec{ab} = \vec{b} - \vec{a}$ and stops

Part (b)	**20 (10, 10) marks**	**Att (3, 3)**

(b) In the triangle *abc*, *p* is a point on the side $[bc]$.

The point *q* lies outside the triangle such that $\vec{pq} = \vec{pb} + \vec{pc} - \vec{pa}$.

(i) Express $\vec{q}$ in terms of $\vec{a}$, $\vec{b}$ and $\vec{c}$.

(ii) Hence show that *abqc* is a parallelogram.

(b) (i) **10 marks** **Att 3**

2 (b) (i)

$$\vec{pq} = \vec{pb} + \vec{pc} - \vec{pa} \Rightarrow \vec{q} - \vec{p} = \vec{b} - \vec{p} + \vec{c} - \vec{p} - \vec{a} + \vec{p}.$$

$$\therefore \vec{q} = \vec{b} + \vec{c} - \vec{a}.$$

Blunders (-3)

B1 $\vec{pq}$ or equivalent expressed incorrectly

Slips (-1)

S1 Arithmetic error

Attempts (3 marks)

A1 $\vec{pq}$ or equivalent expressed correctly

(b) (ii) **10 marks** **Att 3**

2 (b) (ii)

By part (i): $\vec{q} = \vec{b} + \vec{c} - \vec{a} \Rightarrow \vec{q} - \vec{b} = \vec{c} - \vec{a} \Rightarrow \vec{bq} = \vec{ac}.$

$\therefore$ *abqc* is a parallelogram.

Blunders (-3)

B1 $\vec{c} - \vec{a} \neq \vec{ac}$

B2 $\vec{q} - \vec{b} \neq \vec{bq}$

B3 No conclusion or incorrect conclusion

Slips (-1)

S1 Arithmetic error

Attempts (3 marks)

A1 $\vec{q} - \vec{b} = \vec{c} - \vec{a}$

Part (c) **20 (10, 10) marks** **Att (3, 3)**

(c) (i) $\vec{p} = 12\vec{i} + 5\vec{j}$ and $\vec{q} = 3\vec{i} + 4\vec{j}$.

Find the value of the scalar k such that

$$k\left|\vec{p}^{\perp} - \vec{q}\right| = \left|\vec{p}^{\perp}\right| - \left|\vec{q}\right|.$$

(ii) Prove that for all vectors $\vec{r}$ and $\vec{s}$

$$\left(\vec{r} - \vec{s}\right)^{\perp} = \vec{r}^{\perp} - \vec{s}^{\perp}.$$

Part (c) (i) **10 marks** **Att 3**

2 (c) (i)

$$k\left|\vec{p}^{\perp}-\vec{q}\right| = \left|\vec{p}^{\perp}\right| - \left|\vec{q}\right| \Rightarrow k\left|-5\vec{i}+12\vec{j}-3\vec{i}-4\vec{j}\right| = \left|-5\vec{i}+12\vec{j}\right| - \left|3\vec{i}+4\vec{j}\right|.$$

$$\therefore k\left|-8\vec{i}+8\vec{j}\right| = 13-5 \Rightarrow \sqrt{128}k = 8 \Rightarrow 8\sqrt{2}k = 8 \Rightarrow k = \frac{1}{\sqrt{2}} \Rightarrow k = \frac{\sqrt{2}}{2}.$$

Blunders (-3)

B1 $\vec{p}^{\perp}$ incorrect
B2 Error in formula for norm of vector
B3 k not in surd form

Slips (-1)

S1 Arithmetic error

Attempts (3 marks)

A1 Norm of $\vec{q}$
A2 $\vec{p}^{\perp}$ only

Part (c) (ii) **10 marks** **Att 3**

2 (c) (ii)

Let $\vec{r} = a\vec{i}+b\vec{j}$ and $\vec{s} = c\vec{i}+d\vec{j}$. $\therefore \vec{r}-\vec{s} = (a-c)\vec{i}+(b-d)\vec{j}$.

$$\left(\vec{r}-\vec{s}\right)^{\perp} = -(b-d)\vec{i}+(a-c)\vec{j}$$

$$\vec{r}^{\perp}-\vec{s}^{\perp} = -b\vec{i}+a\vec{j}-\left(-d\vec{i}+c\vec{j}\right) = -(b-d)\vec{i}+(a-c)\vec{j} = \left(\vec{r}-\vec{s}\right)^{\perp}.$$

Blunders (-3)

B1 $\vec{r}^{\perp}$ incorrect
B2 No conclusion or incorrect conclusion

Slips (-1)

S1 Arithmetic error

Attempts (3 marks)

A1 One related perpendicular correct
A2 $\vec{r}$ - $\vec{s}$ expressed in terms of $\vec{i}$ and $\vec{j}$
A3 Numerical values for $\vec{r}$ and $\vec{s}$ fully worked out ‘correctly’.

QUESTION 3

Part (a)	**10 marks**	**Att 3**
Part (b)	**20 (10, 10) marks**	**Att (3, 3)**
Part (c)	**20 (5, 5, 5, 5) marks**	**Att (2, 2, 2, 2)**

Part (a) 10 marks Att 3

3 (a)

Find the equation of the line that contains the point $(1, 0)$ and passes through the point of intersection of the lines $2x - y + 6 = 0$ and $10x + 3y - 2 = 0$.

Part (a) 10 marks Att 3

3 (a)

$$6x - 3y + 18 = 0$$
$$10x + 3y - 2 = 0$$
$$16x + 16 = 0 \quad \Rightarrow \quad x = -1 \text{ and } y = 4.$$
$$(1, 0) \text{ and } (-1, 4) \Rightarrow m = \frac{0-4}{1+1} = -2.$$
$$\therefore \text{ Equation of line}: y - 0 = -2(x - 1) \quad \Rightarrow \quad 2x + y - 2 = 0.$$

Blunders (-3)
B1 Error in slope formula
B2 Error in equation of line formula

Slips (-1)
S1 Arithmetic error

Attempts (3 marks)
A1 One co-ordinate of point of intersection
A2 $2x - y + 6 + \lambda(10x + 3y - 2) = 0$

Part (b) 20 (10, 10) marks Att (3, 3)

(b) (i) Prove that the measure of one of the angles between two lines with slopes m_1 and m_2 is given by

$$\tan\theta = \frac{m_1 - m_2}{1 + m_1 m_2}.$$

(ii) Find the equations of the two lines that pass through the point $(6, 1)$ and make an angle of 45° with the line $x + 2y = 0$.

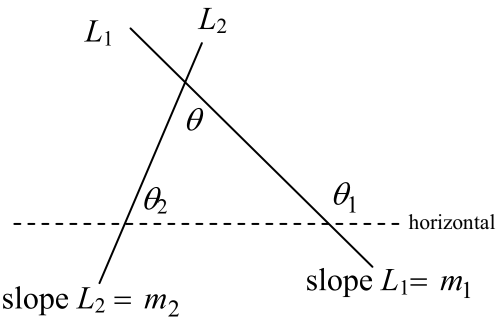
Part (b) (i) **10 marks** **Att 3**

3 (b) (i)

Slope $L_1 = m_1$ and slope $L_2 = m_2$.

Let θ_1 and θ_2 be the positive angles made by L_1 and L_2 respectively with the positive sense of the x-axis.

Then $\tan\theta_1 = m_1$ and $\tan\theta_2 = m_2$

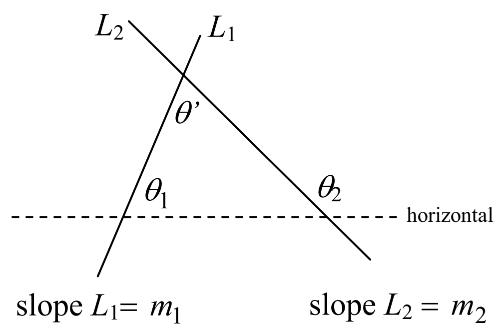


Case 1: ($\theta_1 > \theta_2$)

$$\theta_1 = \theta + \theta_2 \quad \Rightarrow \quad \theta = \theta_1 - \theta_2.$$

$$\tan\theta = \tan(\theta_1 - \theta_2) = \frac{\tan\theta_1 - \tan\theta_2}{1 + \tan\theta_1 \tan\theta_2}$$

$$\therefore \tan\theta = \frac{m_1 - m_2}{1 + m_1 m_2}.$$



Case 2: ($\theta_1 < \theta_2$)

$$\theta_2 = \theta' + \theta_1 \quad \Rightarrow \quad \theta' = -(\theta_1 - \theta_2)$$

$$\tan\theta' = -\tan(\theta_1 - \theta_2) = \frac{\tan\theta_1 - \tan\theta_2}{1 + \tan\theta_1 \tan\theta_2}$$

$$= -\frac{m_1 - m_2}{1 + m_1 m_2}.$$

In this case, the other angle between the lines is $\theta = 180° - \theta'$, giving $\tan\theta = -\tan\theta'$.

* One case to be accepted for full marks

Blunders (-3)

B1 Error in expressing θ in terms of θ_1 and θ_2

B2 Error in expansion of $Tan(\theta_1 - \theta_2)$

Slips (-1)

S1 Arithmetic error

Attempts (3 marks)

A1 $\theta_1 = \theta + \theta_2$ and stops

Part (b) (ii) **10 marks** **Att 3**

3 (b) (ii)

$x + 2y = 0$ has slope $= -\frac{1}{2}$.

$\tan 45^\circ = \pm \frac{m_1 - m_2}{1 + m_1 m_2}$, where $m_2 = -\frac{1}{2}$.

$\therefore\ 1 = \pm \frac{m_1 + \frac{1}{2}}{1 - \frac{1}{2} m_1} \Rightarrow 2 - m_1 = \pm(2m_1 + 1)$

$2 - m_1 = 2m_1 + 1 \Rightarrow m_1 = \frac{1}{3}$ or $2 - m_1 = -2m_1 - 1 \Rightarrow m_1 = -3$.

$y - 1 = \frac{1}{3}(x - 6)$ and $y - 1 = -3(x - 6)$

$x - 3y = 3$ and $3x + y = 19$.

Blunders (-3)
B1 Error in slope
B2 Product of slopes $\neq$ -1
B3 One equation only

Slips (-1)
S1 Arithmetic error

Attempts (3 marks)
A1 Slope of x +2y = 0
A2 *Tan* $45^0 = 1$

Part (c) **20 (5, 5, 5, 5) marks** **Att (2, 2, 2, 2)**

(c) *f* is the transformation $(x, y) \to (x', y')$, where $x' = -x + 2y$ and $y' = 2x - y$.

(i) L is the line $ax + by + c = 0$. Prove that $f(L)$ is a line.

(ii) The line $y = mx$ is its own image under *f*.
Find the two possible values of *m*.

(c) (i) *x* and *y* in terms of x' and y'	**5 marks**	**Att 2**
Substitution	**5 marks**	**Att 2**
Finish	**5 marks**	**Att 2**

3 (c) (i)

$$x' = -x + 2y$$
$$\underline{2y' = 4x - 2y}$$
$$x' + 2y' = 3x \quad \Leftrightarrow \quad x = \frac{1}{3}(x' + 2y').$$
$$y = 2x - y' \quad \Rightarrow \quad y = \frac{2}{3}(x' + 2y') - y' \quad \Rightarrow \quad y = \frac{1}{3}(2x' + y').$$

($\therefore$ The inverse relation is a function and so *f* is clearly bijective $\Pi_0 \to \Pi_0$.)

The set $f(L)$ is the set of all points (x', y') for which $(x, y) \in L$.

$$ax + by + c = 0$$
$$\Leftrightarrow \quad \frac{a}{3}(x' + 2y') + \frac{b}{3}(2x' + y') + c = 0$$
$$\Leftrightarrow \quad (a + 2b)x' + (2a + b)y' + 3c = 0.$$

$\therefore f(L)$ is a line, (since it consists of the set of all points satisfying an equation of the form $px + qy + r = 0$).

OR

(c) (i) Apply *f* to vector form	**5 marks**	**Att 2**
Substitution	**5 marks**	**Att 2**
Finish	**5 marks**	**Att 2**

L is the set $\{\vec{c} + t\vec{m} \mid t \in \mathbf{R}\}$, where $\vec{c} = \begin{pmatrix} 0 \\ -c/b \end{pmatrix}$ and $\vec{m} = \begin{pmatrix} -b \\ a \end{pmatrix}$.

$\therefore f(L)$ is the set $\{f(\vec{c} + t\vec{m}) \mid t \in \mathbf{R}\}$

$= \{f(\vec{c}) + tf(\vec{m}) \mid t \in \mathbf{R}\}$, since *f* is linear.

This is a line, since $f(\vec{m}) \neq \vec{0}$, (as $\det(f) = -3 \neq 0 \Rightarrow f$ is invertible).

Blunders (-3)
B1 $f(L)$ not in the form $px + qy + r = 0$

Slips (-1)
S1 Arithmetic error

Attempts (2,2,2 marks)
A1 Effort at *x* or *y* expressed in terms of x' and y'

(c) (ii)	**5 marks**	**Att 2**

3 (c) (ii)

$(1,\ m) \in y = mx$ and $f(1, m) = (-1 + 2m, 2 - m)$, $f(0, 0) = (0, 0)$.

$\therefore \frac{m}{1} = \frac{2 - m}{-1 + 2m}$ as slope of line and slope of image line are equal.

$\therefore -m + 2m^2 = 2 - m \quad \Rightarrow \quad 2m^2 = 2 \quad \Rightarrow \quad m = \pm 1.$

OR

(c) (ii)	**5 marks**	**Att 2**

$y = mx \quad \Leftrightarrow \quad mx - y + 0 = 0$, so $a = m, b = -1, c = 0$.

So, from part **(i)**, the image is $(m - 2)x' + (2m - 1)y' + 0 = 0$

Rearrange: $y' = \frac{-m + 2}{2m - 1}x'$

This is the same line as $y = mx$, so $\frac{-m + 2}{2m - 1} = m$.

$\therefore -m + 2m^2 = 2 - m \quad \Rightarrow \quad 2m^2 = 2 \quad \Rightarrow \quad m = \pm 1.$

Blunders (-3)

B1 Error in $f(1, m)$ or equivalent

B2 One value of m only

Slips (-1)

S1 Arithmetic error

Attempts (2 marks)

A1 Correct image of any point

A2 Equation of $y = mx$ under f

QUESTION 4

Part (a)	**10 marks**	**Att 3**
Part (b)	**20 (15, 5) marks**	**Att (5, 2)**
Part (c)	**20 (10, 5, 5) marks**	**Att (3, 2, 2)**

Part (a) **10 marks** **Att 3**

Show $(\text{Cos}\,\theta + \text{Sin}\,\theta)^2 + (\text{Cos}\,\theta - \text{Sin}\,\theta)^2 = 2$

Part (a) **10 marks** **Att 3**

4 (a)

$$(\cos\theta+\sin\theta)^2+(\cos\theta-\sin\theta)^2 = \cos^2\theta+2\cos\theta\sin\theta+\sin^2\theta+\cos^2\theta-2\cos\theta\sin\theta+\sin^2\theta$$
$$= 2(\cos^2\theta+\sin^2\theta) = 2.$$

Blunders (-3)

B1 Error in squaring
B2 $\text{Cos}^2\,\theta + \text{Sin}^2\,\theta \neq 1$
B3 Incorrect conclusion

Slips (-1)

S1 Arithmetic error

Attempts (3 marks)

A1 One expansion correct
A2 Verification fully correct

Part (b) **20 (15, 5) marks** **Att (5, 2)**

(b) The lengths of the sides of a triangle are 21, 17 and 10.
The smallest angle in the triangle is A.

(i) Show that $\cos A = \dfrac{15}{17}$.

(ii) Without evaluating A, find $\tan\dfrac{A}{2}$.

(b) (i) **15 marks** **Att 5**

4 (b) (i) The smallest angle is opposite the smallest side, so take $a = 10$.

$$\cos A = \frac{b^2+c^2-a^2}{2bc}$$

$$\cos A = \frac{21^2+17^2-10^2}{2(21)(17)} = \frac{441+289-100}{714} = \frac{630}{714} = \frac{15}{17}.$$

Blunders (-3)

B1 Error in Cosine formula
B2 Error in substitution

Slips (-1)

S1 Arithmetic error

Attempts (5 marks)

A1 Some values substituted into Cosine formula
A2 *Cosine* A expressed in terms of the sides of triangle

(b) (ii) **5 marks** **Att 2**

4 (b) (ii)

$$\cos A = \frac{15}{17} = \frac{1-\tan^2\left(\frac{A}{2}\right)}{1+\tan^2\left(\frac{A}{2}\right)} \Rightarrow 15 + 15\tan^2\left(\frac{A}{2}\right) = 17 - 17\tan^2\left(\frac{A}{2}\right).$$

$$\therefore\ 32\tan^2\left(\frac{A}{2}\right) = 2 \Rightarrow \tan^2\left(\frac{A}{2}\right) = \frac{1}{16} \Rightarrow \tan\frac{A}{2} = \frac{1}{4}$$, (positive, since $0 < \frac{A}{2} < 90°$).

OR

(b) (ii) **5 marks** **Att 2**

4(b)(ii)

$$\cos^2 A = \frac{1}{2}(1 + \cos 2A)$$

$$\cos^2\frac{A}{2} = \frac{1}{2}(1 + \cos A)$$

$$= \frac{1}{2}(1 + \frac{15}{17}) = \frac{16}{17}$$

$$\cos\frac{A}{2} = \pm\frac{4}{\sqrt{17}}.$$

But $0 < \frac{A}{2} < \frac{\pi}{2}$, so $\cos\frac{A}{2} = \frac{4}{\sqrt{17}} = \frac{adj}{hypt}$ in a right angled triangle

$$(\sqrt{17})^2 = 4^2 + opp^2 \Rightarrow opp = 1$$

$$\Rightarrow \tan\frac{A}{2} = \frac{1}{4}$$

Blunders (-3)

B1 Error in formula

B2 $\tan\frac{A}{2}$ negative

Slips (-1)

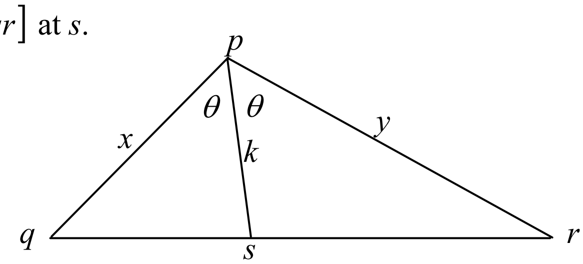
S1 Arithmetic error

Attempts (2 marks)

A1 $\tan\frac{A}{2}$ substituted correctly

Part (c)	**20 (10, 5, 5) marks**	**Att (3, 2, 2)**

(c) The bisector of $\angle qpr$ meets $[qr]$ at s.
$|\angle qpr| = 2\theta$, $|pq| = x$,
$|pr| = y$ and $|ps| = k$.



(i) Find the area of the triangle pqs in terms of x, k and θ.

(ii) Show that $k = \dfrac{2xy\cos\theta}{x+y}$.

(c) (i)	**10 marks**	**Att 3**

4 (c) (i)

Area triangle $pqs = \dfrac{1}{2}xk\sin\theta$.

Blunders (-3)
B1 Area not in required form
B2 $\dfrac{1}{2}$ omitted in formula

Slips (-1)
S1 Arithmetic error

Attempts (3 marks)
A1 Area = $\dfrac{1}{2}$ product of two sides × sine of included angle, with some substitution

(c) (ii) Set up equation	**5 marks**	**Att 2**
Finish	**5 marks**	**Att 2**

4 (c) (ii)

Area triangle pqr = area triangle pqs + area triangle psr.

$$\therefore\ \frac{1}{2}xy\sin 2\theta = \frac{1}{2}xk\sin\theta + \frac{1}{2}ky\sin\theta.$$

$$\Rightarrow\ 2xy\sin\theta\cos\theta = k\sin\theta(x+y)\ \Rightarrow\ k = \frac{2xy\cos\theta}{x+y}.$$

Blunders (-3)
B1 $Sin2\theta$ expanded incorrectly
B2 Error in factors
B3 k not in required form
B4 No conclusion or no conclusion

Slips (-1)
S1 Arithmetic error

Attempts (2, 2 marks)
A1 Area of triangle pqr

QUESTION 5

Part (a)	**10 marks**	**Att 3**
Part (b)	**20 (5, 5, 5, 5) marks**	**Att (2, 2, 2, 2)**
Part (c)	**20 (5, 5, 5, 5) marks**	**Att (2, 2, 2, 2)**

Part (a) **10 marks** **Att 3**

5 **(a)** Find all the solutions of the equation $\cos^2 x - \cos x = 0$, where $0° \leq x \leq 180°$.

5 (a) $\cos^2 x - \cos x = 0 \Rightarrow \cos x(\cos x - 1) = 0$

$\cos x = 0 \Rightarrow x = 90^0$ or $\cos x = 1 \Rightarrow x = 0^0$

Solution is $\{ 0^0, 90^0 \}$

Blunders (-3)
B1 Incorrect factors
B2 Each incorrect value
B3 Each omitted value or 'extra' value

Slips (-1)
S1 Arithmetic error

Attempts (3 marks)
A1 *Cos x* = 0 or *Cos x* -1 = 0

Part (b) **20 (5, 5, 5, 5) marks** **Att (2, 2, 2, 2)**

(b) The function $f : x \to \sin^{-1} x$ is defined for $-1 \leq x \leq 1$.

(i) Copy and complete the table of values of *f* below.

x	-1	$-\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	1
$f(x)$			$-\frac{\pi}{6}$				

(ii) Draw the graph of $y = f(x)$ on graph paper, noting that $\frac{\sqrt{3}}{2} \approx 0{\cdot}87$.

Scale the *y*-axis in terms of π.

(iii) State, with reason, whether each of the following statements is true.

A: "If $\sin x_1 = \sin x_2$, then $x_1 = x_2$".

B: "If $\sin^{-1} x_1 = \sin^{-1} x_2$, then $x_1 = x_2$".

(b) (i)	**5 marks**	**Att 2**

5 (b) (i)

x	-1	$-\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	1
$f(x)$	$-\frac{\pi}{2}$	$-\frac{\pi}{3}$	$-\frac{\pi}{6}$	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$

Slips (-1)

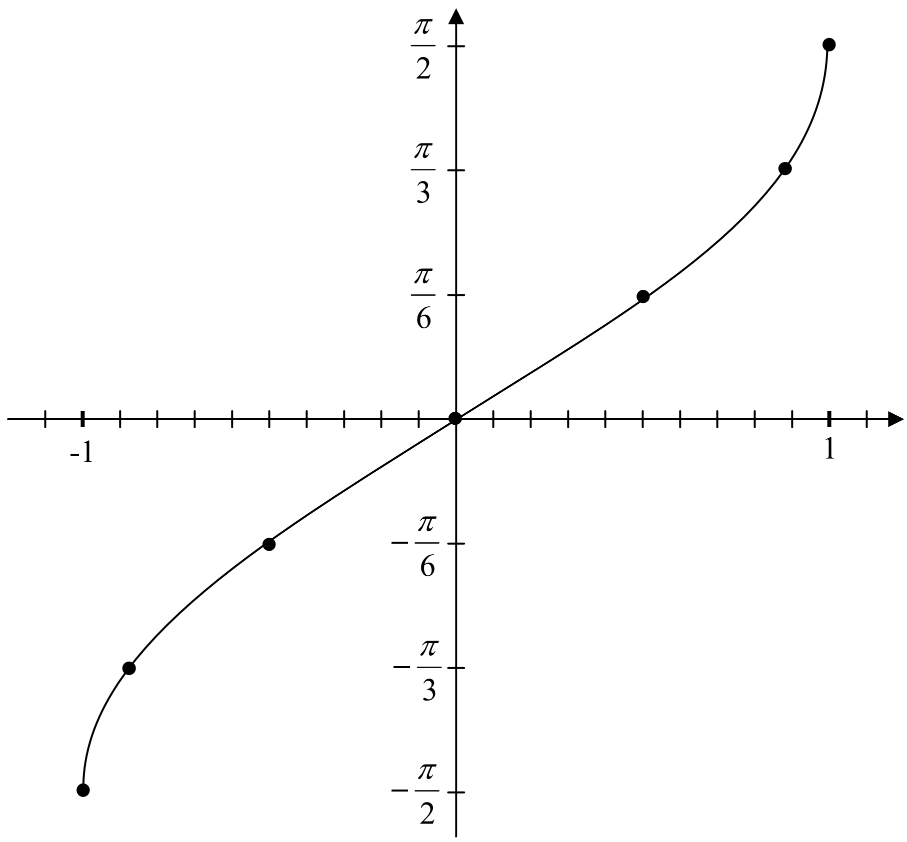
S1 Each incorrect entry to max of 3

Attempts (2 marks)

A1 One correct entry

(b) (ii)	**5 marks**	**Att 2**

5 (b) (ii)



Blunders (-3)

B1 x axis scaled in terms of π (instead of y axis)

B2 Error in scales

B3 Not joining points

Slips (-1)

S1 Each incorrect plot to max of 3.

Attempts (2 marks)

A1 Axes with some correct scale

A2 One point correctly indicated

(b) (iii) A	**5 marks**	**Att 2**
B	**5 marks**	**Att 2**

5 (b) (iii)

A is False:

For example, $\sin 150^{\circ} = \sin 30^{\circ}$, while $150° \neq 30°$

or

A horizontal line can cut the graph of $y = \sin(x)$ more than once.

B is True:

A horizontal line can't cut the graph of $y = \sin^{-1} x$ more than once.

or

$\sin^{-1}$ is strictly increasing on its domain

or

$\sin^{-1}$ is bijective

Blunders (-3)
B1 Correct answer no reason given
B2 Correct answer, incorrect reason

Slips (-1)
S1 Arithmetic error

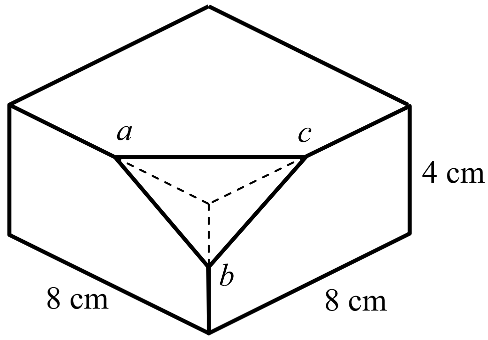
Attempts (2, 2 marks)
A1 $\sin 150^{0} = \sin 30^{0}$ or equivalent

Part (c)	20 (5, 5, 5, 5) marks	Att (2, 2, 2, 2)

(c) A rectangular block of cheese measures 8 cm × 8 cm × 4 cm.

One corner is cut away from the block, in such a way that three of the edges are cut through their midpoints *a*, *b* and *c*.

Find the area of the triangular face *abc* created by the cut.



(c) |*ab*| or |*bc*|	**5 marks**	**Att 2**
|*ac*|	**5 marks**	**Att 2**
Cos	**5 marks**	**Att 2**
Finish	**5 marks**	**Att 2**

5 (c)

$$|ab|^2 = 4^2 + 2^2 \;\Rightarrow\; |ab| = |bc| = \sqrt{20}.$$

$$|ac|^2 = 4^2 + 4^2 \;\Rightarrow\; |ac| = \sqrt{32}.$$

$$\cos\angle abc = \frac{|ab|^2 + |bc|^2 - |ac|^2}{2|ab||bc|} = \frac{20+20-32}{40} = \frac{8}{40} = \frac{1}{5}.$$

$$\therefore \sin\angle abc = \frac{\sqrt{24}}{5} = \frac{2\sqrt{6}}{5}$$

$$\Rightarrow \text{ area triangle } abc = \frac{1}{2}|ab|.|bc|\sin\angle abc = \frac{1}{2}\left(\sqrt{20}\right)\left(\sqrt{20}\right)\frac{2\sqrt{6}}{5} = 4\sqrt{6}\text{ cm}^2.$$

OR

(c) |*ab*| or |*bc*|	**5 marks**	**Att 2**
|*ac*|	**5 marks**	**Att 2**
h	**5 marks**	**Att 2**
Finish	**5 marks**	**Att 2**

abc is an isosceles triangle.

Taking $|ac| = \sqrt{32}$ as base, let *h* be perpendicular height .

$$\therefore \; (\sqrt{20})^2 = h^2 + (\frac{1}{2}\sqrt{32})^2 \quad \Rightarrow h^2 = 12 \Rightarrow \Rightarrow h = 2\sqrt{3}$$

$$\text{Area } \Delta abc = \frac{1}{2}\sqrt{32}.2\sqrt{3} = \sqrt{96} = 4\sqrt{6} \text{ cm}^2$$

Blunders (-3)

B1 Pythagoras incorrect
B2 Incorrect substitution into cosine formula
B3 Incorrect area formula
B4 Area not calculated
B5 sin *A* incorrectly evaluated from cos *A*

Slips (-1)
S1 Arithmetic error
S2 Units omitted

Attempts (2, 2, 2, 2 marks)
A1 Incorrect use of Pythagoras
A2 Some substitution into cosine formula

QUESTION 6

Part (a)	**10 (5, 5) marks**	**Att (2, 2)**
Part (b)	**20 (5, 5, 5, 5) marks**	**Att (2, 2, 2, 2)**
Part (c)	**20 (5, 5, 10) marks**	**Att (2, 2, 3)**

Part (a) **10 (5, 5) marks** **Att (2, 2)**

(a) A student taking a literature course has to read three novels from a list of ten novels.

(i) How many different selections of three novels are possible?

(ii) Two of the ten novels are by the same author. How many selections are possible if the student wishes to choose three novels by different authors?

(a) (i) **5 marks** **Att 2**

6 (a) (i)

Number of selections $^{10}C_3 = 120$

Blunders (-3)

B1 $10 \times 9 \times 8$

Attempts(2 marks)

A1 $^{10}C_x \;,\; x \in \mathbf{N}$

(a) (ii) **5 marks** **Att 2**

6 (a) (ii) Number of selections $= {}^2C_1 \times {}^8C_2 + {}^8C_3 = 56 + 56 = 112.$

or $^{10}C_3 - {}^8C_1 = 120 - 8 = 112$

or $^9C_3 + {}^8C_2 = 84 + 28 = 112$

Blunders (-3)

B1 $^2C_1$ or equivalent missing

B2 $^2C_1 \times {}^8C_2 \times {}^8C_3$

B3 $^2C_1 \times {}^8C_2$

B4 $^8C_1$ or $^9C_3$

Attempts (2 marks)

A1 $^8C_2$ or $^8C_3$ or $^8C_1$

Part (b) **20 (5, 5, 5, 5) marks** **Att (2, 2, 2, 2)**

(b)	**(i)**	In how many different ways can eight people be seated in a row?
	(ii)	Three girls and five boys sit in a row, arranged at random. Find the probability that the three girls are seated together.
	(iii)	Three girls and n boys sit in a row, arranged at random. If the probability that the three girls are seated together is $\frac{1}{35}$, find the value of n.

Part (b) (i) **5 marks** **Att 2**

6 (b) (i)

Number of ways = 8!.

Blunders (-3)

B1 $^{8}C_{8}$ or 8^8

Slips (-1)

S1 Arithmetic error

Attempts (2 marks)

A1 8 + 7 + 6 + 5 + 4 + 3 + 2 + 1

Part (b) (ii) **5 marks** **Att 2**

6 (b) (ii)

Number of possible arrangements = 8!.
Number of favourable arrangements = 6!×3!.

$$\text{Probability} = \frac{6!\times 3!}{8!} = \frac{6}{56} = \frac{3}{28}.$$

Blunders (-3)

B1 Incorrect number of possible outcomes
B2 Incorrect number of favourable outcomes (e.g. 5! . 3!)
B3 6! + 3!
B4 6! x 3
B5 No divisor

Slips (-1)

S1 Arithmetic error

Attempts (2 marks)

A1 Correct number of favourable outcomes
A2 Correct number of possible outcomes

Part (b) (iii) Probability	**5 marks**	**Att 2**
Finish	**5 marks**	**Att 2**

6 (b) (iii)

Number of possible arrangements $= (n+3)!$.

Number of favourable arrangements $= (n+1)! \times 3!$.

$\therefore$ Probability $= \dfrac{(n+1)! \times 3!}{(n+3)!} = \dfrac{6}{(n+3)(n+2)}$.

$\therefore \dfrac{6}{(n+3)(n+2)} = \dfrac{1}{35}$.

$\therefore n^2 + 5n + 6 = 210 \Rightarrow n^2 + 5n - 204 = 0$.

$(n-12)(n+17) = 0 \quad \Rightarrow \quad n = 12$, as $n \neq -17$.

Solution is $n = 12$.

Blunders (-3)

B1 Incorrect number of possible outcomes
B2 Incorrect number of favourable outcomes
B3 Error in simplifying factorials
B4 Error in factors of quadratic equation

Slips (-1)

S1 Arithmetic error
S2 $n = -17$ not excluded

Attempts (2,2 marks)

A1 Correct number of possible outcomes
A2 Correct number of favourable outcomes
A3 Use of $n + 1$

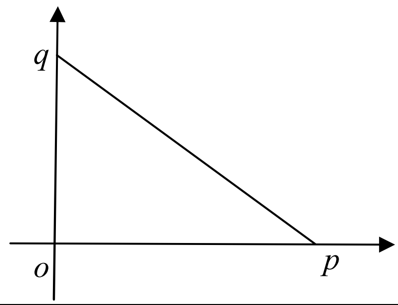
Part (c) **20 (5, 5, 10) marks** **Att (2, 2, 3)**

(c) x and y are randomly selected integers with $1 \le x \le 10$ and $1 \le y \le 10$.
p is the point with coordinates $(x, 0)$ and q is the point with coordinates $(0, y)$.
Find the probability that

(i) the slope of pq is equal to -1

(ii) the slope of pq is greater than -1

(iii) the length of $[pq]$ is less than or equal to 5.



(c) (i) **5 marks** **Att 2**
(ii) **5 marks** **Att 2**
(iii) **10 marks** **Att 3**

(i) $○ = \frac{10}{100} = \frac{1}{10}$

(ii) $■ = \frac{45}{100} = \frac{9}{20}$

(iii) $✓ = \frac{15}{100} = \frac{3}{20}$

y \ x	1	2	3	4	5	6	7	8	9	10
1	○✓	■✓	■✓	■✓	■	■	■	■	■	■
2	✓	○✓	■✓	■✓	■	■	■	■	■	■
3	✓	✓	○✓	■✓	■	■	■	■	■	■
4	✓	✓	✓	○	■	■	■	■	■	■
5					○	■	■	■	■	■
6						○	■	■	■	■
7							○	■	■	■
8								○	■	■
9									○	■
10										○

OR

Part (c) (i) **5 marks** **Att 2**

6 (c) (i)

Slope $pq = \frac{y-0}{0-x} = -\frac{y}{x}$ and $-\frac{y}{x} = -1$, when $x = y$.

$\therefore$ Number of favourable outcomes is 10.

Number of possible outcomes is $10 \times 10 = 100$.

$\therefore$ Probability $= \frac{10}{100} = \frac{1}{10}$.

Blunders (-3)

B1 $y \neq x$ implied

B2 Incorrect number of possible outcomes

B3 Incorrect number of favourable outcomes

Slips (-1)

S1 Arithmetic error

Attempts (2 marks)
A1 Listing some favourable outcomes
A2 Listing total number of outcomes

Part (c) (ii) **5 marks** **Att 2**

6 (c) (ii)

$$pq = \frac{y-0}{0-x} = -\frac{y}{x}. \quad -\frac{y}{x} > -1 \Rightarrow \frac{y}{x} < 1 \Rightarrow y < x.$$

$\therefore$ Number of favourable outcomes is 9+8+7+6+5+4+3+2+1 = 45.
Number of possible outcomes is 10×10 = 100.
$\therefore$ Probability $= \frac{45}{100} = \frac{9}{20}$.

Blunders (-3)
B1 $y < x$ not implied
B2 Incorrect number of possible outcomes
B3 Incorrect number of favourable outcomes

Slips (-1)
S1 Arithmetic error

Attempts (2 marks)
A1 Listing favourable outcomes
A2 Listing total number of outcomes
A3 Drawing a grid with some relevant items

Part (c) (iii) **10 marks** **Att 3**

6 (c) (iii)

$$|pq| = \sqrt{x^2+y^2}. \quad |pq| \le 5 \Rightarrow x^2+y^2 \le 25.$$

$\therefore$ Favourable outcomes are
$x \in \{1, 2, 3, 4\}$ and $y \in \{1, 2, 3, 4\}$ but with $x = 4$ and $y = 4$ not included.
$\therefore$ Number of favourable outcomes is $(4\times4)-1 = 15$..
Number of possible outcomes is 10×10 = 100.
$\therefore$ Probability $= \frac{15}{100} = \frac{3}{20}$.

Blunders (-3)
B1 $x^2 + y^2 \le 25$ or equivalent not implied
B2 Incorrect number of possible outcomes
B3 Incorrect number of favourable outcomes

Slips (-1)
S1 Arithmetic error

Attempts (3 marks)
A1 Listing some favourable outcomes
A2 Listing total number of outcomes
A3 Some use of Pythagoras

QUESTION 7

Part (a)	**10 marks**	**Att 3**
Part (b)	**20 (5, 5, 5, 5) marks**	**Att (2, 2, 2, 2)**
Part (c)	**20 (5, 5, 5, 5) marks**	**Att (2, 2, 2, 2)**

Part (a) **10 marks** **Att 3**

7 **(a)** The prices of four food items in a shopping basket are €3, €5, €1 and €6. Find the weighted mean price of these items using the weights 2, 3, 4 and 1 respectively.

Part (a) **10 marks** **Att 3**

7 (a)

$$\text{Weighted mean } = \frac{2(3)+3(5)+4(1)+1(6)}{10} = \frac{31}{10} = €3{\cdot}10.$$

Blunders (-3)
B1 Sum of weights incorrect
B2 Incorrect denominator
B3 $x + w$ instead of xw for each term

Slips (-1)
S1 Arithmetic error
S2 Omission of currency symbol

Attempts (3 marks)
A1 Sum of weights

Part (b) **20 (5, 5, 5, 5) marks** **Att (2, 2, 2, 2)**

(b) **(i)** Solve the difference equation $u_{n+2} - 6u_{n+1} + 5u_n = 0$, where $n \geq 1$, given that $u_1 = 0$ and $u_2 = 20$.

(ii) Find an expression in n for the sum of the terms $u_1 + u_2 + u_3 + ... + u_n$.

Part (b) (i)	**Characteristic Equation**	**5 marks**	**Att 2**
	$\boldsymbol{u_n}$	**5 marks**	**Att 2**
	Finish	**5 marks**	**Att 2**

7 (b) (i)

$$u_{n+2} - 6u_{n+1} + 5u_n = 0 \;\Rightarrow\; x^2 - 6x + 5 = 0.$$

$$\therefore\; (x-1)(x-5) = 0 \;\Rightarrow\; x = 1 \text{ or } x = 5.$$

$$u_n = p(\alpha)^n + q(\beta)^n = p(1)^n + q(5)^n \;\Rightarrow\; u_n = p + q(5)^n.$$

$$u_1 = p + 5q = 0 \text{ and } u_2 = p + 25q = 20.$$

$$\therefore\; 20q = 20 \Rightarrow q = 1 \text{ and hence } p = -5.$$

$$\therefore\; u_n = -5 + 5^n.$$

Blunders (-3)
B1 Error in setting up quadratic
B2 Error in solving quadratic
B3 Error in general term
B4 Error in finding *p* and *q*

Slips (-1)
S1 Arithmetic error

Attempts (2, 2,2 marks)
A1 Substitution into quadratic formula
A2 Attempt at finding *p* or *q*

Part (b) (ii) **5 marks** **Att 2**

7 (b) (ii)

$$u_1 + u_2 + u_3 + ... + u_n = \sum u_n = -5n + \sum_{n=1}^{n} 5^n$$

$$= -5n + \frac{5(5^n - 1)}{5-1} = -5n + \frac{5}{4}(5^n - 1).$$

Blunders (-3)
B1 Error in forming geometric series
B2 Error in sum of geometric series
B3 Mishandling -5

Slips (-1)
S1 Arithmetic error

Attempts (2 marks)
A1 Using formula for sum to infinity of G.P.
A2 Correct formula with some substitution
A3 Listing at least 3 consecutive terms correctly

Part (c)	**20 (5, 5, 5, 5) marks**	**Att (2, 2, 2, 2)**

(c) The two numbers a and b have mean $\bar{x}$ an and standard deviation σ_1.
The three numbers c, d and e have mean $\bar{x}$ and standard deviation σ_2.
Find the standard deviation of the five numbers a, b, c, d and e in terms of σ_1 and σ_2.

Part (c) Expressions for σ_1 and σ_2	**5 marks**	**Att 2**
Both $\bar{x}$ for (*a,b* and *c,d,e*)	**5 marks**	**Att 2**
Mean *a,b,c,d,e*	**5 marks**	**Att 2**
Finish	**5 marks**	**Att 2**

7 (c)

$$\frac{a+b}{2}=\bar{x}, \quad \frac{c+d+e}{3}=\bar{x}.$$

$$\sigma_1=\sqrt{\frac{(a-\bar{x})^2+(b-\bar{x})^2}{2}}, \quad \sigma_2=\sqrt{\frac{(c-\bar{x})^2+(d-\bar{x})^2+(e-\bar{x})^2}{3}}.$$

Mean of $a, b, c, d, e = \frac{a+b+c+d+e}{5}=\frac{2\bar{x}+3\bar{x}}{5}=\bar{x}.$

Standard deviation of $a, b, c, d, e = \sqrt{\frac{(a-\bar{x})^2+(b-\bar{x})^2+(c-\bar{x})^2+(d-\bar{x})^2+(e-\bar{x})^2}{5}}$

$$=\sqrt{\frac{2\sigma_1^{\,2}+3\sigma_2^{\,2}}{5}}.$$

OR

Part (c) Expressions for σ_1 and σ_2	**5 marks**	**Att 2**
Both $\bar{x}$ (for *a, b* and *c, d, e*)	**5 marks**	**Att 2**
Mean *a, b, c, d, e*	**5 marks**	**Att 2**
Finish	**5 marks**	**Att 2**

$$\sum\frac{x^2}{n}-(\bar{x})^2 \Rightarrow \sigma_1^{\,2}=\frac{a^2+b^2}{2} - \bar{x}^2 \text{ and } \sigma_2^{\,2}=\frac{c^2+d \ +e^2}{3} - \bar{x}^2$$

But $\frac{a+b}{2}=\bar{x}$ and $\frac{c+d+e}{3}=\bar{x} \Rightarrow \frac{a+b+c+d+e}{5}=\frac{2\bar{x}+3\bar{x}}{5}=\bar{x}$

$$\sigma^2=\frac{a^2+b^2+c^2+d^2+e^2}{5}-\bar{x}^2$$

$$=\frac{a^2+b^2+c^2+d^2+e^2-5\bar{x}^2}{5}$$

$$=\frac{a^2+b^2-2\bar{x}^2+c^2+d^2+e^2-3\bar{x}^2}{5}$$

$$=\frac{2\sigma_1^{\,2}+3\sigma_2^{\,2}}{5}$$

$$\therefore \ \sigma=\sqrt{\frac{2\sigma_1^{\,2}+3\sigma_2^{\,2}}{5}}$$

Blunders (-3)

B1 Error in mean

B2 Error in standard deviation

Slips (-1)

S1 Arithmetic error

Attempts (2, 2, 2, 2 marks)

A1 Correct mean of *a* and *b*

A2 One correct standard deviation

A3 Expression for mean of *a, b, c, d, e*

QUESTION 8

Part (a)	**15 (5, 5, 5) marks**	**Att (2, 2, 2)**
Part (b)	**20 (15, 5) marks**	**Att (5, 2)**
Part (c)	**15 (5, 5, 5) marks**	**Att (2, 2, 2)**

Part (a)	**15 (5, 5, 5) marks**	**Att (2, 2, 2)**

8. **(a)** Use integration by parts to find $\int xe^{4x}dx$.

Part (a) Assign parts	**5 marks**	**Att 2**
$\frac{du}{dx}$ **and** ***v***	**5 marks**	**Att 2**
Finish	**5 marks**	**Att 2**

8 (a)

$$\int xe^{4x}dx = uv - \int vdu, \text{ where } u = x \Rightarrow du = dx \text{ and } dv = e^{4x}dx \Rightarrow v = \frac{1}{4}e^{4x}.$$

$$\therefore \int xe^{4x}dx = \frac{1}{4}xe^{4x} - \int \frac{1}{4}e^{4x}dx = \frac{1}{4}xe^{4x} - \frac{1}{16}e^{4x} + c = \frac{e^{4x}}{16}(4x-1) + c.$$

Blunders (-3)
B1 Incorrect differentiation or integration
B2 Incorrect ‘parts’ formula

Slips (-1)
S1 Arithmetic error
S2 Omits constant of integration

Attempts (2,2,2 marks)
A1 One correct assigning to parts formula
A2 Correct differentiation or integration

Part (b)	**20 (15, 5) marks**	**Att (5, 2)**

(b)	**(i)**	Derive the first four terms of the Maclaurin series for $f(x)=\sqrt{1+x}$.
	(ii)	Given that this series converges for $-1<x<1$, use these four terms to find an approximation for $\sqrt{17}$, as a fraction.

(b) (i)	**15 marks**	**Att 5**

8 (b) (i)

$$f(x)=f(0)+f'(0)x+\frac{f''(0)x^2}{2!}+\frac{f'''(0)x^3}{3!}+........$$

$$f(x)=\sqrt{1+x} \quad \Rightarrow \quad f(0)=1.$$

$$f'(x)=\frac{1}{2}(1+x)^{-\frac{1}{2}} \quad \Rightarrow \quad f'(0)=\frac{1}{2}.$$

$$f''(x)=-\frac{1}{4}(1+x)^{-\frac{3}{2}} \quad \Rightarrow \quad f''(0)=-\frac{1}{4}.$$

$$f'''(x)=\frac{3}{8}(1+x)^{-\frac{5}{2}} \quad \Rightarrow \quad f'''(0)=\frac{3}{8}.$$

$$\therefore\ f(x)=1+\frac{x}{2}-\frac{x^2}{8}+\frac{x^3}{16}+...........$$

Blunders (-3)
B1 Incorrect differentiation
B2 Incorrect evaluation of $f^{(n)}(0)$
B3 Each term not derived
B4 Error in Maclaurin series

Slips(-1)
S1 Arithmetic error

Attempts (5 marks)
A1 Correct expansion for *f(x)* given but not derived
A2 *f(0)* correct
A3 A correct differentiation
A4 Any one correct term

(b) (ii) **5 marks** **Att 2**

8 (b) (ii)

$$\sqrt{17} = \sqrt{16+1} = 4\sqrt{1+\frac{1}{16}} = 4\sqrt{1+x}, \text{ for } x = \frac{1}{16}.$$

$$\therefore \sqrt{17} = 4\left[1+\frac{1}{32}-\frac{1}{2048}+\frac{1}{65536}+...\right] = 4\left[\frac{67553}{65536}\right] = \frac{67553}{16384}.$$

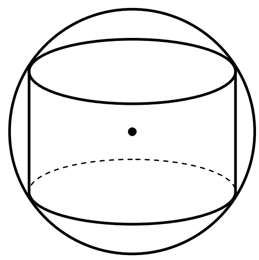
Blunders (-3)
B1 Mishandling of $\sqrt{16+1}$
B2 Answer not in form $\frac{a}{b}$, $a \in Z$, $b \in Z$

Slips (-1)
S1 Arithmetic error

Attepts(2 marks)
A1 17 as sum of 16 and 1 or 17 as sum of 9 and 8
A2 Answer in decimal form with relevant work

Part (c) **15 (5, 5, 5) marks** **Att (2, 2, 2)**

(c) The diagram shows a cylinder inscribed in a sphere.
The cylinder has height $2x$ and radius r.
The sphere has fixed radius a.



(i) Express r in terms of a and x.

(ii) Find, in terms of a, the maximum possible volume of the cylinder.

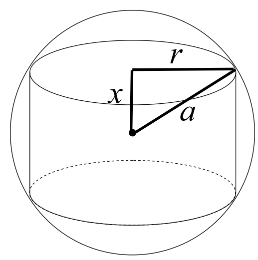
(c) (i) **5 marks** **Att 2**

8 (c) (i)

$$r^2 + x^2 = a^2$$

$$r^2 = a^2 - x^2$$

$$\Rightarrow \quad r = \sqrt{a^2 - x^2}$$



Blunders (-3)
B1 Error in Pythagoras
B2 Incorrect side in triangle

Slips (-1)
S1 Arithmetic error

Attempts (2 marks)
A1 $a^2 = r^2 + x^2$

(c)(ii) Volume in terms of *x* **5 marks** **Att 2**
Finish **5 marks** **Att 2**

8 (c) (ii)

Volume of cylinder $= V = \pi r^2 h.$

$\therefore\ V = \pi\left(a^2 - x^2\right)2x = 2\pi a^2 x - 2\pi x^3.$

$\frac{dV}{dh} = 2\pi a^2 - 6\pi x^2 = 0$ for max or min. $\Rightarrow\ x = \frac{a}{\sqrt{3}}.$

$\frac{d^2V}{dh^2} = -12\pi x < 0,$ for $x = \frac{a}{\sqrt{3}}$

$\Rightarrow$ maximum volume at $x = \frac{a}{\sqrt{3}}.$

$\therefore\ V = \pi\left(a^2 - \frac{a^2}{3}\right)\frac{2a}{\sqrt{3}} = \frac{4\pi a^3}{3\sqrt{3}} = \frac{4\sqrt{3}\pi a^3}{9}.$

* $\frac{d^2V}{dh^2}$ not required

Blunders (-3)
B1 Error in differentiation
B2 Error in finding x
B3 Error in indices

Slips (-1)
S1 Arithmetic error

Attempts (2, 2 marks)
A1 Some part of correct substitution into volume
A2 Some correct differentiation
A3 $\frac{dV}{dx} = 0$ indicated for max or min

QUESTION 9

Part (a)	**10 (5, 5) marks**	**Att (2, 2)**
Part (b)	**20 (5, 5, 5, 5) marks**	**Att (2, 2, 2, 2)**
Part (c)	**20 (5, 5, 5, 5) marks**	**Att (2, 2, 2, 2)**

Part (a) **10 marks** **Att 3**

9. (a) A and B are independent events such that $P(A) = 0{\cdot}25$ and $P(A \cup B) = 0{\cdot}55$.
Find $P(B)$.

Part (a) Apply independence rule **5 marks** **Att 2**
Finish **5 marks** **Att 2**

9 (a)

$$P(A \cup B) = P(A) + P(B) - P(A \cap B).$$
$$\therefore P(A \cup B) = P(A) + P(B) - P(A).P(B).$$
$$\therefore \; 0{\cdot}55 = 0{\cdot}25 + P(B) - (0{\cdot}25)P(B) \Rightarrow 0{\cdot}75P(B) = 0{\cdot}3.$$
$$\therefore \; P(B) = 0{\cdot}4.$$

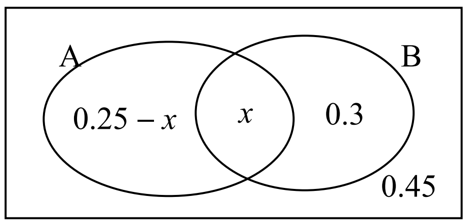
OR

A and B independent $\Leftrightarrow$ A’ and B’ independent.

$$P(A')P(B') = P(A' \cap B') = P((A \cup B)')$$
$$0.75P(B') = 0.45$$
$$P(B') = 0.6$$
$$P(B) = 0.4$$

OR

$$P(A \cap B) = P(A)P(B)$$
$$x = (0.25)(0.3 + x)$$
$$= 0.075 + 0.25x$$
$$0.75x = 0.075$$
$$x = 0.1$$
$$P(B) = 0.4$$



Blunders (-3)
B1 $P(A \cap B) \neq P(A).P(B)$
B2 $P(A \cup B) = P(A) + P(B) + P(A).P(B)$

Slips (-1)
S1 Arithmetic error

Attempts (2,2 marks)
A1 $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

Part (b) **20 (5, 5, 5, 5) marks** **Att (2, 2, 2, 2)**

(b) A person plays a game that involves throwing five hoops at a peg. The following table gives the probability distribution for the number of hoops that land on the peg.

x	0	1	2	3	4	5
$P(x)$	0·01	0·08	0·23	0·34	0·26	0·08

Find the mean and the standard deviation of the distribution.

Part (b) Mean	**5 marks**	**Att 2**
Deviations	**5 marks**	**Att 2**
$(x-\bar{x})^2 . P(x)$ **expressed**	**5 marks**	**Att 2**
Finish	**5 marks**	**Att 2**

9 (b)

$$\text{Mean} = \bar{x} = \sum_{x=0}^{5} xP(x) = 0{\cdot}08 + 0{\cdot}46 + 1{\cdot}02 + 1{\cdot}04 + 0{\cdot}4 = 3.$$

$$\text{Standard deviation} = \sigma = \sqrt{\sum_{x=0}^{5} (x-\bar{x})^2 . P(x)}.$$

$$= \sqrt{(0-3)^2(0{\cdot}01) + (1-3)^2(0{\cdot}08) + (2-3)^2(0{\cdot}23) + (3-3)^2(0{\cdot}34) + (4-3)^2(0{\cdot}26) + (5-3)^2(0{\cdot}08)}$$

$$\therefore\ \sigma = \sqrt{0{\cdot}09 + 0{\cdot}32 + 0{\cdot}23 + 0 + 0{\cdot}26 + 0{\cdot}32} = \sqrt{1{\cdot}22}.$$

Blunders (-3)

B1 $\sum P(x)$ incorrect

B2 $\sum x$ denominator for mean

B3 Use of $\sum (x + P(x))$

B4 Mishandles deviation

B5 Incorrect standard deviation formula

Slips (-1)

S1 Arithmetic error

Attempts (2,2,2,2 marks)

A1 Any correct *x.P(x)*

A2 Correct formula with some substitution

A3 Any correct deviation

Part (c) **20 (5, 5, 5, 5) marks** **Att (2, 2, 2, 2)**

(c)	A coin is slightly bent and is thought to favour heads. Accordingly, it is tossed 100 times to test the null hypothesis that it is fair against the alternative hypothesis that it favours heads. In this experiment, 55 heads are observed. **(i)** Show that this result is not significant at the 5% level. **(ii)** How many times would the coin have to be tossed in an experiment in order that an observation of 55% heads *would* be regarded as significant at the 5% level?

(c) (i) σ **5 marks** **Att 2**

Finish **5 marks** **Att 2**

9 (c) (i)

$H_\circ$: coin is fair.

This is a one tailed test.

$p = \frac{1}{2},\ q = \frac{1}{2},\ n = 100. \quad \bar{x} = np = 50,\ \sigma = \sqrt{npq} = 5.$

$z = \frac{x - \bar{x}}{\sigma} = \frac{55 - 50}{5} = 1.$

$z = 1 < 1{\cdot}645.$

$\therefore$ The observed result is not significant at the 5% level.

Blunders (-3)

B1 Incorrect value of *p* or of *q*
B2 Incorrect formula for mean
B3 Incorrect formula for standard deviation
B4 Error in standard units
B5 Two tailed test
B6 Misreads tables
B7 Incorrect conclusion

Slips (-1)

S1 Arithmetic error

Attempts (2,2 marks)

A1 Correct value for *p* or *q*
A2 Correct formula for mean with some substitution
A3 Correct formula for standard deviation with some substitution
A4 Correct expression for standard units with some substitution

(c) (ii) Value of *z*	**5 marks**	**Att 2**
Finish	**5 marks**	**Att 2**

9 (c) (ii)

$$\bar{x} = np = \frac{n}{2}, \quad \sigma = \sqrt{npq} = \frac{\sqrt{n}}{2}.$$

$$z = \frac{\frac{55n}{100} - \frac{n}{2}}{\frac{\sqrt{n}}{2}} = \frac{\frac{n}{10}}{\sqrt{n}} = \frac{\sqrt{n}}{10}.$$

$$\frac{\sqrt{n}}{10} > 1{\cdot}645 \Rightarrow \sqrt{n} > 16{\cdot}45 \Rightarrow n > 270{\cdot}6.$$

$\therefore$ 271 times

Blunders (-3)

B1 Incorrect value of p or of q
B2 Incorrect formula for mean
B3 Incorrect formula for standard deviation
B4 Error in standard units
B5 Mishandles 55%
B6 Incorrect conclusion

Slips (-1)

S1 Arithmetic error
S2 Stops at $n > 270.6$

Attempts (2,2 marks)

A1 Correct value for p or q
A2 Some substitution for standard units

QUESTION 10

Part (a)	10 marks	Att 3
Part (b)	20 (5, 5, 5, 5) marks	Att (2, 2, 2, 2)
Part (c)	20 (5, 5, 5, 5) marks	Att (2, 2, 2, 2)

Part (a) **10 marks** **Att 3**

10. **(a)** If a is the permutation $\begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}$, find $a \circ a$.

10 (a)

$$a \circ a = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}.$$

Blunders (-3)
B1 Each incorrect element (max. of 2)

Slips (-1)
S1 Arithmetic error

Attempts (3 marks)
A1 Permutation incomplete
A2 One element correct with another repeated

Part (b) **20 (5, 5, 5, 5) marks** **Att (2, 2, 2, 2)**

(b) The set {1, 2, 4, 5, 7, 8} is a group under multiplication modulo 9.

(i) Draw up a Cayley table for the group.

(ii) Find a generator of the group.

(iii) Hence, or otherwise, find a subgroup of order 2 and a subgroup of order 3.

(b) (i) **5 marks** **Att 2**

10 (b) (i)

× mod 9	1	2	4	5	7	8
1	1	2	4	5	7	8
2	2	4	8	1	5	7
4	4	8	7	2	1	5
5	5	1	2	7	8	4
7	7	5	1	8	4	2
8	8	7	5	4	2	1

Blunders (-3)
B1 Not closed

Slips (-1)
S1 Arithmetic error
S2 Each incorrect entry to max of 3

Attempts (2 marks)
A1 Incomplete table

(b) (ii)	**5 marks**	**Att 2**

10 (b) (ii)

5 is a generator. $5^1 = 5,\ 5^2 = 7,\ 5^3 = 8,\ 5^4 = 4,\ 5^5 = 2,\ 5^6 = 1.$

or 2 is also a generator $2^1 = 2, 2^2 = 4, 2^3 = 8, 2^4 = 7, 2^5 = 5, 2^6 = 1$

Blunders (-3)
B1 Error in verifying generator
B2 Identity or other element not shown in terms of generator

Attempts (2marks)
A1 Generator identified but not demonstrated for any element
A2 Attempts to establish a generator

Part (b) (iii)	**10 (5, 5) marks**	**Att (2, 2)**

10 (b) (iii)

Subgroup of order 2 is {1, 8}.
Subgroup of order 3 is {1, 4, 7}

Blunders (-3)
B1 Incorrect element in subgroup

Slips (-1)
S1 Arithmetic error

Attempts (2, 2 marks)
A1 Identity only correct element

Part (c) **20 (5, 5, 5, 5) marks** **Att (2, 2, 2, 2)**

(c) $(G, \circ)$ and $(H, *)$ are two groups with identities e_G and e_H respectively. If $\phi : G \to H$ is an isomorphism, prove that **(i)** $\phi(e_G) = e_H$. **(ii)** $\phi(x^{-1}) = [\phi(x)]^{-1}$, for all $x \in G$.

(c) (i) Establish correspondence	**5 marks**	**Att 2**
Finish	**5 marks**	**Att 2**

10 (c) (i)

Let $x \in G$.

$\phi(x \circ e_G) = \phi(x) * \phi(e_G)$, because of isomorphism.

$\phi(x) = \phi(x) * \phi(e_G)$, as $x \circ e_G = x$.

$\Rightarrow \phi(e_G)$ is the identity in $(H, *)$.

$\therefore \phi(e_G) = e_H$.

Blunders (-3)

B1 Error in setting up correspondence in operators

B2 No conclusion

Slips (-1)

S1 Arithmetic error

Attempts (2,2 marks)

A1 $x \circ e_G = x$

(c) (ii) Establish correspondence	**5 marks**	**Att 2**
Finish	**5 marks**	**Att 2**

10 (c) (ii)

$\phi(x \circ x^{-1}) = \phi(x) * \phi(x^{-1})$.

$\therefore \phi(e_G) = \phi(x) * \phi(x^{-1}) \Rightarrow \phi(x)$ and $\phi(x^{-1})$ are inverses.

$\therefore \phi(x^{-1}) = [\phi(x)]^{-1}$.

Blunders (-3)

B1 $x \circ x^{-1} \neq e_G$ or equivalent

B2 Not indicating $\phi(x)$ *and* $\phi(x^{-1})$ are inverses

B3 No conclusion

Slips (-1)

S1 Arithmetic error

Attempts (2,2 marks)

A1 $x \circ x^{-1} = e$

QUESTION 11

Part (a)	**10 marks**	**Att 3**
Part (b)	**20 (5, 5, 5, 5) marks**	**Att (2, 2, 2, 2)**
Part (c)	**20 (5, 5, 5, 5) marks**	**Att (2, 2, 2, 2)**

Part (a) **10 marks** **Att 3**

(a) Find the equation of the ellipse with foci $(\sqrt{7},\ 0)$ and $(-\sqrt{7},\ 0)$ and with eccentricity $\frac{\sqrt{7}}{4}$.

11 (a)

$$e = \frac{\sqrt{7}}{4} \text{ and } ae = \sqrt{7} \Rightarrow a = 4.$$

$$b^2 = a^2(1 - e^2) = 16\left(1 - \frac{7}{16}\right) = 9.$$

$$\therefore \text{ ellipse}: \frac{x^2}{16} + \frac{y^2}{9} = 1.$$

Blunders (-3)
B1 Formula error

Slips (-1)
S1 Arithmetic error

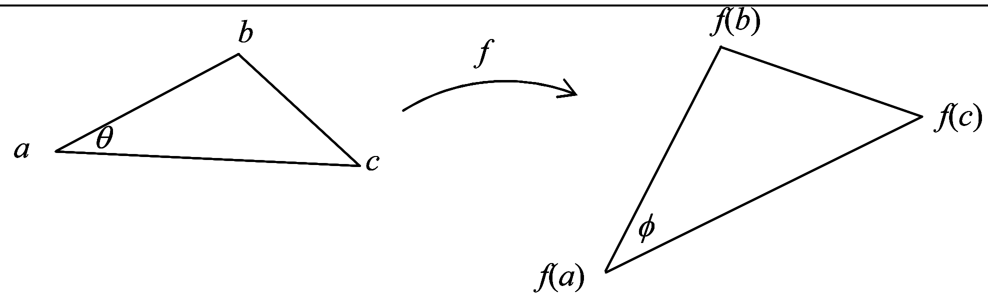
Attempts (3 marks)
A1 $ae = \sqrt{7}$ and stops

Part (b) **20 (5, 5, 5, 5) marks** **Att (2, 2, 2, 2)**

(b)	A transformation f is a similarity transformation if there exists a fixed number k such that $\left|f(a)f(b)\right| = k\left|ab\right|$, for all a and b. Show that angle measure is invariant under a similarity transformation.

Part (b) f defined	**5 marks**	**Att 2**
$\cos\angle\phi$	**5 marks**	**Att 2**
Use of k^2	**5 marks**	**Att 2**
Finish	**5 marks**	**Att 2**

11 (b)



The triangle abc is mapped onto the triangle $f(a)f(b)f(c)$ under a similarity transformation f. $\angle\theta$ is mapped onto $\angle\phi$.

$$\cos\angle\phi = \frac{|f(a)f(b)|^2 + |f(a)f(c)|^2 - |f(b)f(c)|^2}{2|f(a)f(b)|.|f(a)f(c)|} = \frac{k^2|ab|^2 + k^2|ac|^2 - k^2|bc|^2}{2k^2|ab||ac|}.$$

$$\therefore\ \cos\angle\phi = \frac{|ab|^2 + |ac|^2 - |bc|^2}{2|ab|.|ac|} = \cos\angle bac = \cos\angle\theta.$$

$\therefore\ |\angle\theta| = |\angle\phi|$, since both are in the range $0°$ to $180°$

Hence angle measure is invariant under a similarly transformation.

Blunders (-3)
B1 Error in cosine formula
B2 Fails to identify $|f(a)f(b)|=k|ab|$ or equivalent
B3 No conclusion or incorrect conclusion

Slips (-1)
S1 Arithmetic error

Attempts (2, 2, 2, 2 marks)
A1 *Cos ø* with some substitution

Part (c) **20 (5, 5, 5, 5) marks** **Att (2, 2, 2, 2)**

(c)	**(i)**	Define the term *conjugate diameters* of an ellipse.
	(ii)	Prove that all parallelograms circumscribed to a given ellipse at the endpoints of conjugate diameters have the same area.

(c) (i)	**5 marks**	**Att 2**

11 (c) (i)	If $\|pq\|$ is a diameter of an ellipse E, then there is a second diameter $\|rs\|$, such that $\|pq\|$ bisects all chords of E on lines parallel to $\|rs\|$ and vice versa. $\|pq\|$ and $\|rs\|$ are called *conjugate diameters* of the ellipse.

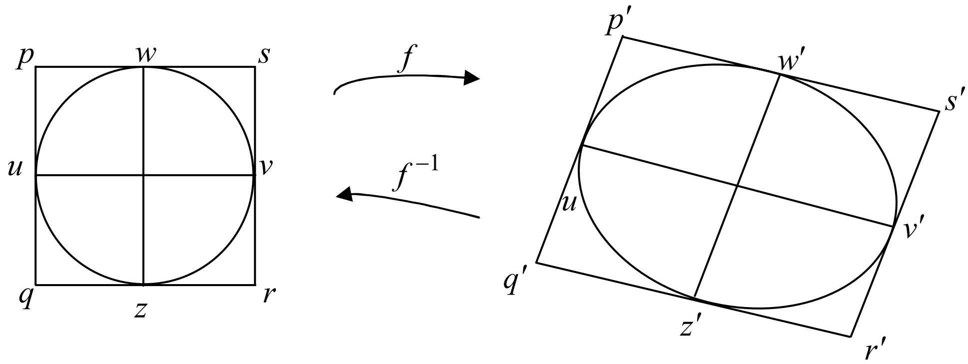
Blunders (-3)
B1 Parallel property not indicated

Slips (-1)
S1 Arithmetic error

Attempts (2 marks)
A1 Incomplete relevant diagram

Part (c) (ii) f	**5 marks**	**Att 2**
f^{-1}	**5 marks**	**Att 2**
Finish	**5 marks**	**Att 2**

11 (c) (ii)



$[u'v']$ and $[w'z']$ are conjugate diameters of ellipse E.
Tangents at their end points form the parallelogram $p'q'r's'$.
Under an affine transformation f^{-1}, the ellipse maps to the circle $x^2 + y^2 = 1$ and $p'q'r's'$ is mapped to $pqrs$.
$[uv]$ and $[wz]$ are conjugate diameters of the circle and $uv \perp wz$. The square $pqrs$ has fixed area 4 sq units.
$\therefore$ Area $p'q'r's' = |\text{det.}(f)|.\text{area } pqrs = 4\det(f)$.
But det(f) is constant $\Rightarrow$ area $p'q'r's'$ is constant.
$\therefore$ Areas of all parallelograms at end points of conjugate diameters of an ellipse are equal.

Blunders (-3)
B1 Fails to identify conjugate diameters of circle are perpendicular
B2 Fails to identify det(f) is constant and / or area $pqrs$ is constant

Slips (-1)
S1 Arithmetic error

Attempts (2, 2, 2 marks)
A1 Some relevant mapping

MARCANNA BREISE AS UCHT FREAGAIRT TRÍ GHAEILGE

(Bonus marks for answering through Irish)

Ba chóir marcanna de réir an ghnáthráta a bhronnadh ar iarrthóirí nach ngnóthaíonn níos mó ná 75% d'iomlán na marcanna don pháipéar. Ba chóir freisin an marc bónais sin a shlánú **síos**.

Déantar an cinneadh agus an ríomhaireacht faoin marc bónais i gcás gach páipéir ar leithligh.

Is é 5% an gnáthráta agus is é 300 iomlán na marcanna don pháipéar. Mar sin, bain úsáid as an ngnáthráta 5% i gcás iarrthóirí a ghnóthaíonn 225 marc nó níos lú, e.g. 198 marc × 5% = 9·9 ⇒ bónas = 9 marc.

Má ghnóthaíonn an t-iarrthóir níos mó ná 225 marc, ríomhtar an bónas de réir na foirmle [300 – bunmharc] × 15%, agus an marc bónais sin a shlánú **síos**. In ionad an ríomhaireacht sin a dhéanamh, is féidir úsáid a bhaint as an tábla thíos.

Bunmharc	Marc Bónais
226	11
227 – 233	10
234 – 240	9
241 – 246	8
247 – 253	7
254 – 260	6
261 – 266	5
267 – 273	4
274 – 280	3
281 – 286	2
287 – 293	1
294 – 300	0